

# EXAMPLES IN DEPENDENT THEORIES

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**ABSTRACT.** In the first part we show a counterexample to a conjecture by Shelah regarding the existence of indiscernible sequences in dependent theories (up to the first inaccessible cardinal). In the second part we discuss generic pairs, and give an example where the pair is not dependent. Then we define the notion of directionality which deals with counting the number of co-heirs of a type, and prove a trichotomy theorem about it. Finally we discuss non-splintering, an interesting notion that appears in the work of Rami Grossberg, Andrés Villaveces and Monica VanDieren, and we show that it is not trivial whenever the directionality of the theory is not small.

## 1. INTRODUCTION

**1.1. Existence of indiscernibles.** In the summer of 2008, Saharon Shelah announced in a talk in Rutgers that he had proved some very important results in dependent theories. One of these was the existence of indiscernible, an old conjuncture of his.

Recall,

**Definition 1.1.** A first order theory  $T$  is *dependent* (NIP) if it does not have the independence property which means: there is no formula  $\varphi(x, y)$  and tuples  $\langle a_i, b_s \mid i < \omega, s \subseteq \omega \rangle$  in  $\mathfrak{C}$  such that  $\varphi(a_i, b_s)$  if and only if  $i \in s$ .

When we say that  $T$  has existence of indiscernibles we mean that inside a large enough set, one can always find a sequence of indiscernible. To make things more precise, we use the following notation:

**Definition 1.2.** Let  $T$  be a theory. For a cardinal  $\kappa$ ,  $n \leq \omega$  and an ordinal  $\delta$ ,  $\kappa \rightarrow (\delta)_{T,n}$  means: for every set  $A \subseteq \mathfrak{C}^n$  of size  $\kappa$ , there is a non-constant sequence of elements of  $A$  of length  $\delta$  which is indiscernible.

In stable theories, it is known that for any  $\lambda$  satisfying  $\lambda = \lambda^{|T|}$ ,  $\lambda^+ \rightarrow (\lambda^+)_{T,n}$  (proved by Shelah in [She90], and follows from local character of non-forking). The first claim of this kind was proved by Morley in [Mor65] for  $\omega$ -stable theories. In [She05], Shelah proved:

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**Fact 1.3.** *If  $T$  is strongly dependent (see Definition 2.3 below), then for all  $\lambda \geq |T|$ ,  $\beth_{|T|}(\lambda) \rightarrow (\lambda^+)_{T,n}$  for all  $n < \omega$ .*

This definition was suggested by Grossberg and Shelah and appears in [She86, pg. 208, Definition 3.1(2)], in a slightly different form<sup>1</sup>. There it is also proved that such a theorem does not hold for simple unstable theories in general. So the natural generalization is to NIP theories, and indeed it is conjectured already in [She86, pg. 209, Conjecture 3.3].

This conjecture is connected to a result by Shelah and Cohen: in [CS09], they proved that a theory is stable if and only if it can be presented in some sense in a free algebra in a fix vocabulary but allowing function symbols with infinite arity. If this result could be extended to: a theory is dependent if and only if it can be represented as an algebra with ordering, then this could be used to prove existence of indiscernibles.

Despite announcing it, there was a mistake in the proof for dependent theories, and here we shall see a counter-example. In this paper, we shall show that

**Theorem 1.4.** *There is a countable dependent theory  $T$  such that if  $\kappa$  is smaller than the first inaccessible cardinal, then for all  $n \in \omega$ ,  $\kappa \not\rightarrow (\omega)_{T,n}$ .*

It appears in a more precise way as Theorem 2.11 below.

An even stronger result can be obtained, namely

**Theorem 1.5.** *For every  $\theta$  there is a dependent theory  $T$  of size  $\theta$  such that for all  $\kappa$  and  $\delta$ ,  $\kappa \rightarrow (\delta)_{T,1}$  if and only if  $\kappa \rightarrow (\delta)_{\theta}^{<\omega}$ .*

where

**Definition 1.6.**  $\kappa \rightarrow (\delta)_{\theta}^{<\omega}$  means: for every coloring  $c : [\kappa]^{<\omega} \rightarrow \theta$  there is an homogeneous sub-sequence of length  $\delta$  (i.e. there exists  $\langle \alpha_i \mid i < \delta \rangle \in {}^{\delta}\lambda$  and  $\langle c_n \mid n < \omega \rangle \in {}^{\omega}\theta$  such that  $c(\alpha_{i_0}, \dots, \alpha_{i_{n-1}}) = c_n$  for every  $i_0 < \dots < i_{n-1} < \delta$ ).

One can see that  $\kappa \rightarrow (\delta)_{\theta}^{<\omega}$  always implies that  $\kappa \rightarrow (\delta)_{T,n}$  for all  $n < \omega$ , so this is the best result possible.

However, the proof of Theorem 1.5 is considerably harder, so it is given in a subsequent work [KS10].

The second part of the paper is devoted to giving a related example in the field of real numbers. By Fact 1.3, as  $\text{Th}(\mathbb{R})$  is strongly dependent, we cannot prove the same theorem, but instead we show that the requirement that  $n < \omega$  is necessary:

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<sup>1</sup>The definition there is:  $\kappa \rightarrow (\delta)_{n,T}$  if and only if for each sequence of length  $\kappa$  (of  $n$ -tuples), there is an indiscernible sub-sequence of length  $\delta$ . For us there is no difference because we are dealing with examples where  $\kappa \not\rightarrow (\mu)_{T,n}$ . It is also not hard to see that when  $\delta$  is a cardinal these two definitions are equivalent.

**Theorem 1.7.** *If  $\kappa$  is smaller than the first strongly inaccessible cardinal, then  $\kappa \not\vdash (\omega)_{\text{RCF}, \omega}$ .*

This is Theorem 2.16 below.

*Notes.* Many strong results that have been announced in that talk in 2008 remain true. For instance, the main theorem there is that a theory is dependent if and only if the number of types up to isomorphism is small (see more in [She12]).

We also should note that a related result can be found in an unpublished paper in Russian by Kudajbergenov that states that for every ordinal  $\alpha$  there exists a dependent theory (but it may be even strongly dependent)  $T_\alpha$  such that  $|T_\alpha| = |\alpha| + \aleph_0$  and  $\beth_\alpha(|T_\alpha|) \not\vdash (\aleph_0)_{T_\alpha, 1}$  and thus seem to indicate that the bound in Fact 1.3 is tight.

*Idea of the construction.* The counterexample is a “tree of trees” with functions connecting the different trees. For every  $\eta$  in the tree  ${}^{\omega \geq 2}$  we shall have a predicate  $P_\eta$  and an ordering  $<_\eta$  such that  $(P_\eta, <_\eta)$  is a dense tree. In addition we shall have functions  $G_{\eta, \eta \smallfrown \{i\}} : P_\eta \rightarrow P_{\eta \smallfrown \{i\}}$  for  $i = 0, 1$ . The idea is to prove that  $\kappa \not\vdash (\mu)_{T, 1}$  by induction on  $\kappa$ , i.e. to prove that in  $P_\emptyset$  there are no indiscernibles. To use the induction hypothesis, we push the counter examples we already have for smaller  $\kappa$ ’s to deeper levels in the tree  ${}^{\omega \geq 2}$ .

**1.2. Generic pairs.** In a series of papers ([She, She06, She11, She12]), Shelah has proved (among other things) that dependent theories give rise to a “generic pair” of models (and in fact this characterizes dependent theories). The natural question is whether the theory of the pair is again dependent. The answer is no. We present an example of an  $\omega$ -stable theory all of whose generic pairs have the independence property.

**1.3. Directionality.** The directionality of a theory measures the number of finitely satisfiable global extensions of a complete type (these are also called co-heirs). We say that a theory has *small* directionality if for every type  $p$  over a model  $M$ , the number of complete finitely satisfiable (in  $M$ )  $\Delta$ -types which are consistent with  $p$  is finite for all finite sets  $\Delta$ . The theory has *medium* directionality if this number is bounded by  $|M|$ , and it has *large* directionality if it is not. We prove a trichotomy theorem (Theorem 4.21 below), showing that every theory is either of small, medium or large directionality.

We provide examples of dependent theories of each kind of directionality.

**1.4. Splintering.** The last part of the paper is connected to a paper by the work of Rami Grossberg, Andrés Villaveces and Monica VanDieren. In [GAV] they study Shelah’s Generic pair conjecture (which is now a theorem) and in their analysis, they came up with the notion of splintering which is similar to splitting. We show that in any dependent theory with medium or large directionality, splitting is different than splintering. We also provide an example of such a theory with small directionality, and prove this cannot happen in the stable realm.

**1.5. Notation.** We do not distinguish elements and tuples unless we say so explicitly.

$\mathfrak{C}$  will be the monster model of the theory.

$S(A)$  is the set of all types over  $A$ .

For a set of formulas with a partition of variables,  $\Delta(x, y)$ ,  $L_\Delta(A)$  is the set of formulas of the form  $\varphi(x, a), \neg\varphi(x, a)$  where  $\varphi(x, y) \in \Delta$  and  $a \in A$ .  $S_\Delta(A)$  is the set of all complete  $L_\Delta(A)$ -types over  $A$ . Similarly we may define  $\text{tp}_\Delta(b/A)$  as the set of formulas  $\varphi(x, a)$  such that  $\varphi(x, y) \in \Delta$  and  $\mathfrak{C} \models \varphi(b, a)$ . For a partial type  $p(x)$  over  $A$ ,  $p \upharpoonright \Delta = p \cap L_\Delta(A)$ .

Usually we want to consider a set of formulas  $\Delta$  without specifying a partition of the variables. In this case, for a tuple of variables  $x$ ,  $\Delta^x$  is a set of partitioned formulas induced from  $\Delta$  by partitioning the formulas in  $\Delta$  to  $(x, y)$  in all possible ways. Then  $L_\Delta^x(A)$  is just  $L_{\Delta^x}(A)$  and  $S_\Delta^x(A)$ ,  $p \upharpoonright \Delta^x$  are defined similarly. If  $x$  is clear from the context, we omit it. So for instance, when  $p$  is type in  $x$  over  $A$ , then  $p \upharpoonright \Delta$  is the set of all formulas  $\varphi(x, a)$  where  $\varphi(z, w) \in \Delta$ .

## 2. LACK OF INDISCERNIBLES

### 2.1. Preliminaries.

**Definition 2.1.** We shall need the following fact:

**Fact 2.2.** [She90, II, 4] *Let  $T$  be any theory. Then for all  $n < \omega$ ,  $T$  is dependent if and only if  $\square_n$  if and only if  $\square_1$  where for all  $n < \omega$ ,*

$\square_n$  For every finite set of formulas  $\Delta(x, y)$  with  $n = \text{lg}(x)$ , there is a polynomial  $f$  such that for every finite set  $A \subseteq M \models T$ ,  $|S_\Delta(A)| \leq f(|A|)$ .

Since we also discuss strongly dependent theories, here is the definition:

**Definition 2.3.** A theory is called *strongly dependent* if there is no sequence of formulas

$$\langle \varphi_n(x, y_n) \mid n < \omega \rangle$$

such that the following set of formulas is consistent with the theory

$$\left\{ \varphi_n(x_\eta, y_{n,k})^{\eta(n)=k} \mid \eta : \omega \rightarrow \omega \right\}.$$

Where  $\varphi^{\text{True}} = \varphi$ ,  $\varphi^{\text{False}} = \neg\varphi$ .

See [She05] for further discussion of strongly dependent theories. There it is proved that  $\text{Th}(\mathbb{R})$  is strongly dependent, and so is the theory of the  $p$ -adics.

**2.2. An example of lack of indiscernibles.** Let  $S_n$  be the finite binary tree  ${}^n\geq 2$ . On a well ordered tree such as  $S_n$ , we define  $<_{\text{suc}}$  as follows:  $\eta <_{\text{suc}} \nu$  if  $\nu$  is a successor of  $\eta$  in the tree.

Let  $L_n$  be the following language:

$$L_n = \{P_\eta, <_\eta, \wedge_\eta, G_{\eta,\nu} \mid \eta, \nu \in S_n, \eta <_{\text{suc}} \nu\}.$$

Where:

- $P_\eta$  is a unary predicate ;  $<_\eta$  is a binary relation symbol ;  $\wedge_\eta$  is a binary function symbol ;  $G_{\eta,\nu}$  is a unary function symbol.

Let  $T_n^\forall$  be the following theory:

- $P_\eta \cap P_\nu = \emptyset$  for  $\eta \neq \nu$ .
- $(P_\eta, <_\eta, \wedge_\eta)$  is a tree, where  $\wedge_\eta$  is the meet function on  $P_\eta$ , i.e.

$$x \wedge_\eta y = \max \{z \in P_\eta \mid z \leq_\eta x \ \& \ z \leq_\eta y\}$$

(so if for example,  $x \notin P_\eta$ , then  $x \wedge_\eta y = x$ ).

- $G_{\eta,\nu} : P_\eta \rightarrow P_\nu$ , meaning that outside  $P_\eta$ ,  $G_{\eta,\nu}$  is the identity, and no further restrictions.

Thus we have:

*Claim 2.4.*  $T_n^\forall$  is a universal theorem.

*Claim 2.5.*  $T_n^\forall$  has the joint embedding property (JEP) and the amalgamation property (AP).

*Proof.* Easy to see. □

From this we deduce, by e.g. [Hod93, Theorem 7.4.1]:

**Corollary 2.6.**  $T_n^\forall$  has a model completion,  $T_n$  which eliminates quantifiers, and moreover: if  $M \models T_{n+1}^\forall$ ,  $M' = M \upharpoonright L_n$  and  $M' \subseteq N' \models T_n^\forall$  then  $N'$  can be enriched to a model  $N$  of  $T_{n+1}^\forall$  so that  $M \subseteq N$ . Hence if  $M$  is an existentially closed model of  $T_{n+1}^\forall$ , then  $M'$  is an e.c. model of  $T_n^\forall$ . Hence  $T_n \subseteq T_{n+1}$  (for more see [Hod93, Theorem 8.2.4]).

*Proof.* The moreover part: for each  $\eta \in S_{n+1} \setminus S_n$ , we define  $P_\eta^N = P_\eta^M$  and in the same way  $\wedge_\eta$ . The functions  $G_{\eta,\nu}$  for  $\eta \in S_n$  and  $\nu \in S_{n+1}$  will be extensions of  $G_{\eta,\nu}$ . □

Now we show that  $T_n$  is dependent, but before that, a few easy remarks:

**Observation 2.7.**

- (1) If  $A \subseteq M \models T_0^\forall$  is a finite substructure (so just a tree, with no extra structure), then for all  $b \in M$ , the structure generated by  $A$  and  $b$  is  $A \cup \{b\} \cup \{\max\{b \wedge a \mid a \in A\}\}$ .

- (2) If  $M \models T_n^\forall$  and  $\eta \in {}^{n \geq 2}$ , we can define a new structure  $M_\eta \models T_{n-\lg(\eta)}^\forall$  whose universe is  $\bigcup \left\{ P_{\eta \smallfrown v}^M \mid v \in {}^{n-\lg(\eta) \geq 2} \right\}$  by:  $P_v^{M_\eta} = P_{\eta \smallfrown v}^M$ , and in the same way we interpret every other symbol (for instance,  $G_{v_1, v_2}^{M_\eta} = G_{\eta \smallfrown v_1, \eta \smallfrown v_2}^M$ ). For every formula  $\varphi(x) \in L_{n-\lg(\eta)}$  there is a formula  $\varphi'(x) \in L_n$  such that for all  $a \in M_\eta$ ,  $M \models \varphi'(a)$  if and only if  $M_\eta \models \varphi(a)$  (we get  $\varphi'$  by concatenating  $\eta$  before any symbol).
- (3) For  $M$  as before and  $\eta \in {}^{n \geq 2}$ , for any  $k < \omega$  there is a bijection between

$$\left\{ p(x_0, \dots, x_{k-1}) \in S_k^{\text{qf}}(M) \mid \forall i < k (P_\eta(x_i) \in p) \right\}$$

and

$$\left\{ p(x_0, \dots, x_{k-1}) \in S_k^{\text{qf}}(M_\eta) \mid \forall i < k (P_{\langle \rangle}(x_i) \in p) \right\}.$$

*Proof.* (3): The bijection is given by (2). This is well defined, meaning that if  $p(x_0, \dots, x_{k-1})$  is a type over  $M_\eta$  such that  $P_{\langle \rangle}(x_i) \in p$  for all  $i < k$ , then  $\{\varphi' \mid \varphi \in p\}$  determines a complete type over  $M$ , such that  $P_\eta(x_i) \in p$  for all  $i < k$ . This is because any atomic formula of the form  $F(x, a) <_v G(x, b)$  where  $a, b \in M$  and  $F, G$  are terms is either trivially false, as in the case  $v \not\geq \eta$ , or trivially equivalent to a formula with no involvement of  $x$ 's at all (as in the case  $a \wedge_\eta x <_v b$  where  $a \notin P_\eta$ ) or it is equivalent to a formula of the form  $F'(x, a') <_v G'(x, b')$  where  $a', b'$  are from  $M_\eta$ .  $\square$

**Proposition 2.8.**  $T_n$  is dependent.

*Proof.* We use Fact 2.2. It is sufficient to find a polynomial  $f(x)$  such that for every finite set  $A$ ,  $|S_1(A)| \leq f(|A|)$ .

First we note that for a set  $A$ , the size of the structure generated by  $A$  is bounded by a polynomial in  $|A|$ : it is generated by applying  $\wedge_{\langle \rangle}$  on  $P_{\langle \rangle} \cap A$ , applying  $G_{\langle \rangle, \langle 1 \rangle}$  and  $G_{\langle \rangle, \langle 0 \rangle}$ , and then applying  $\wedge_{\langle 0 \rangle}, \wedge_{\langle 1 \rangle}$  and so on. Every step in the process is polynomial, and it ends after  $n$  steps.

Hence we can assume that  $A$  is a substructure, i.e.  $A \models T_n^\forall$ .

The proof is by induction on  $n$ . To easy notation, we shall omit the subscript  $\eta$  from  $<_\eta$  and  $\wedge_\eta$ .

First we deal with the case  $n = 0$ . In  $T_0$ ,  $P_{\langle \rangle}$  is a tree with no extra structure, while outside  $P_{\langle \rangle}$  there is no structure at all. The number of types outside  $P_{\langle \rangle}$  is bounded by  $|A| + 1$  (because there is only one non-algebraic type). In the case that  $P_{\langle \rangle}(x) \in p$  for some type  $p$  over  $A$ , we can characterize  $p$  by characterizing the type of  $x' := \max\{a \wedge x \mid a \in A\}$ , i.e. the cut that  $x'$  induces on the tree, and by knowing whether  $x' = x$  or  $x > x'$  (we note that in general, every theory of a tree is dependent by [Par82]).

Now assume that the claim is true for  $n$ . Suppose  $\eta \in {}^{n+1 \geq 2}$  and  $1 \leq \lg(\eta)$ . By Observation 2.7(3), there is a bijection between the types  $p(x)$  over  $A$  where  $P_\eta(x) \in p$  and the types  $p(x)$

in  $T_{n+1-\lg(\eta)}$  over  $A_\eta$  where  $P_\langle \rangle \in \mathbf{p}$ .  $A_\eta \models T_{n+1-\lg(\eta)}^\forall$ , and so by the induction hypothesis, the number of types over  $A_\eta$  is bounded by a polynomial in  $|A_\eta| \leq |A|$ . As the number of types  $\mathbf{p}(x)$  such that  $P_\eta(x) \notin \mathbf{p}$  for all  $\eta$  is bounded by  $|A|+1$  as in the previous case, we are left with checking the number of types  $\mathbf{p}(x)$  such that  $P_\langle \rangle(x) \in \mathbf{p}$ .

In order to describe  $\mathbf{p}$ , we first have to describe  $\mathbf{p}$  restricted to the language  $\{<_\langle \rangle, \wedge_\langle \rangle\}$ , and this is polynomially bounded. Let  $x' = \max\{a \wedge x \mid a \in A\}$ . By Observation 2.7(1), if  $A \cup \{x\}$  is not closed under  $\wedge_\langle \rangle$ ,  $x'$  is the only new element in the structure generated by  $A \cup \{x\}$  in  $P_\langle \rangle$ . Hence, we are left to determine the type of the pairs  $(G_{\langle \rangle, \langle i \rangle}(x), G_{\langle \rangle, \langle i \rangle}(x'))$  over  $A$  for  $i = 0, 1$  (if  $x'$  is not new, then it's enough to determine the type of  $G_{\langle \rangle, \langle i \rangle}(x)$ ). The number of these types is equal to the number of types of pairs in  $T_n$  over  $A_{\langle i \rangle}$ . As  $T_n$  is dependent we are done by Fact 2.2.  $\square$

**Definition 2.9.** Let  $L = \bigcup_{n < \omega} L_n$ ,  $T = \bigcup_{n < \omega} T_n$  and  $T^\forall = \bigcup_{n < \omega} T_n^\forall$ .

We easily have

**Corollary 2.10.**  $T$  is complete, it eliminates quantifiers and is dependent.

We shall prove the following theorem:

**Theorem 2.11.** For any two cardinals  $\mu \leq \kappa$  such that in  $[\mu, \kappa]$  there are no (uncountable) strongly inaccessible cardinals,  $\kappa \nrightarrow (\mu)_{T,1}$ .

We shall prove a slightly stronger statement, by induction on  $\kappa$ :

**Proposition 2.12.** Given  $\mu$  and  $\kappa$ , such that either  $\kappa < \mu$  or there are no (uncountable) strongly inaccessible cardinals in  $[\mu, \kappa]$ , there is a model  $M \models T^\forall$  such that  $|P_\langle \rangle^M| \geq \kappa$  and  $P_\langle \rangle^M$  does not contain a non-constant indiscernible sequence (for quantifier free formulas) of length  $\mu$ .

From now on, indiscernible will only mean “indiscernible for quantifier free formulas”.

*Proof.* Fix  $\mu$ . The proof is by induction on  $\kappa$ . We divide into cases:

*Case 1.*  $\kappa < \mu$ . Clear.

*Case 2.*  $\kappa = \mu = \aleph_0$ . Denote  $\eta_j = \langle 1 \dots 1 \rangle$ , i.e. the constant sequence of length  $j$  and value 1. Find  $M \models T^\forall$  such that its universe contains a set  $\{a_{i,j} \mid i, j < \omega\}$  where  $a_{i,j} \neq a_{i',j'}$  for all  $(i,j) \neq (i',j')$ ,  $a_{i,j} \in P_{\eta_j}$  and in addition  $G_{\eta_j, \eta_{j+1}}(a_{i,j}) = a_{i,j+1}$  if  $j < i$  and  $G_{\eta_j, \eta_{j+1}}(a_{i,j}) = a_{0,j+1}$  otherwise. We also need that  $P_\langle \rangle^M = \{a_{i,0} \mid i < \omega\}$ . Any model satisfying these properties will do (so no need to specify what the tree structures are). Now, if in  $P_\langle \rangle^M = \{a_{i,0} \mid i < \omega\}$  there is a non-constant indiscernible sequence,  $\langle a_{i_k,0} \mid k < \omega \rangle$ , then there is a large enough  $j$  such that

$$G_{\eta_j, \eta_{j+1}} \circ \dots \circ G_{\eta_0, \eta_1}(a_{i_0,0}) = G_{\eta_j, \eta_{j+1}} \circ \dots \circ G_{\eta_0, \eta_1}(a_{i_1,0}).$$

But for every large enough  $k$ ,  $G_{\eta_j, \eta_{j+1}} \circ \dots \circ G_{\eta_0, \eta_1} (a_{i_1, 0}) \neq G_{\eta_j, \eta_{j+1}} \circ \dots \circ G_{\eta_0, \eta_1} (a_{i_k, 0})$  — contradiction.

*Case 3.*  $\kappa$  is singular. Suppose  $\kappa = \bigcup_{i < \sigma} \lambda_i$  where  $\sigma, \lambda_i < \kappa$  for all  $i < \sigma$ . By the induction hypothesis, for  $i < \sigma$  there is a model  $M_i \models T^\forall$  such that  $|P_{\langle \rangle}^{M_i}| \geq \lambda_i$  and in  $P_{\langle \rangle}^{M_i}$  there is no non-constant indiscernible sequence of length  $\mu$ . Also, there is a model  $N$  such that  $|P_{\langle \rangle}^N| \geq \sigma$  and in  $P_{\langle \rangle}^N$  there is no non-constant indiscernible sequence of length  $\mu$ . We may assume that the universes of all these models are pairwise disjoint and disjoint from  $\kappa$ .

Suppose that  $\{a_i \mid i < \sigma\} \subseteq P_{\langle \rangle}^N$ , and  $\{b_j \mid \sum_{l < i} \lambda_l \leq j < \lambda_i\} \subseteq P_{\langle \rangle}^{M_i}$  witness that  $|P_{\langle \rangle}^N| \geq \sigma$  and  $|P_{\langle \rangle}^{M_i}| \geq \lambda_i \setminus \sum_{l < i} \lambda_l$ . Let  $\bar{M}$  be a model extending each  $M_i$  and containing the disjoint union of the sets  $\bigcup_{i < \sigma} M_i$  (exists by JEP).

Define a new model  $M \models T^\forall$ :  $(P_{\langle \rangle}^M, <_{\langle \rangle}) = (\kappa, <)$  (so  $\wedge_{\langle \rangle} = \min$ );  $(P_{\langle 1 \rangle \frown \eta}^M, <_{\eta}) = (P_{\eta}^N, <_{\eta})$  and  $(P_{\langle 0 \rangle \frown \eta}^M, <_{\langle 0 \rangle, \eta}) = (P_{\eta}^{\bar{M}}, <_{\eta})$ . In the same way define  $\wedge_{\eta}$  for all  $\eta$  of length  $\geq 1$ . The functions are also defined in the same way:  $G_{\langle 1 \rangle \frown \eta, \langle 1 \rangle \frown \nu}^M = G_{\eta, \nu}^N$  and  $G_{\langle 0 \rangle \frown \eta, \langle 0 \rangle \frown \nu}^M = G_{\eta, \nu}^{\bar{M}}$ . We are left to define  $G_{\langle \rangle, \langle 0 \rangle}$  and  $G_{\langle \rangle, \langle 1 \rangle}$ . So let:  $G_{\langle \rangle, \langle 1 \rangle}(\alpha) = a_{\min\{i \mid \alpha < \lambda_i\}}$  and  $G_{\langle \rangle, \langle 0 \rangle}(\alpha) = b_{\alpha}$  for all  $\alpha < \kappa$ .

Note that if  $I$  is an indiscernible sequence contained in  $P_{\langle 1 \rangle}^M$  then  $I$  is an indiscernible sequence in  $N$  contained in  $P_{\langle \rangle}^N$ , and the same is true for  $P_{\langle 0 \rangle}^M$  and  $\bar{M}$ .

Assume  $\langle \alpha_j \mid j < \mu \rangle$  is an indiscernible sequence in  $P_{\langle \rangle}^M$ . Then  $\langle G_{\langle \rangle, \langle 1 \rangle}(\alpha_j) \mid j < \mu \rangle$  is a constant sequence (by the choice of  $N$ ). So there is  $i < \sigma$  such that  $\sum_{l < i} \lambda_l \leq \alpha_j < \lambda_i$  for all  $j < \mu$ . So  $\langle G_{\langle \rangle, \langle 0 \rangle}(\alpha_j) = b_{\alpha_j} \mid j < \mu \rangle$  is a constant sequence (it is indiscernible in  $P_{\langle \rangle}^{\bar{M}}$  and in fact contained in  $P_{\langle \rangle}^{M_i}$ ), hence  $\langle \alpha_j \mid j < \mu \rangle$  is constant, as we wanted.

*Case 4.*  $\kappa$  is regular uncountable. By the hypothesis of the proposition,  $\kappa$  is not strongly inaccessible, so there is some  $\lambda < \kappa$  such that  $2^\lambda \geq \kappa$ . By the induction hypothesis on  $\lambda$ , there is a model  $N \models T^\forall$  such that in  $P_{\langle \rangle}^N$  there is no non-constant indiscernible sequence of length  $\mu$ . Let  $\{a_i \mid i \leq \lambda\} \subseteq P_{\langle \rangle}^N$  witness that  $|P_{\langle \rangle}^N| \geq \lambda$ .

Define  $M \models T^\forall$  as follows:  $P_{\langle \rangle}^M = {}^{\lambda \geq 2}$  and the ordering is inclusion (equivalently, the ordering is by initial segment).  $\wedge_{\langle \rangle}$  is defined naturally:  $f \wedge_{\langle \rangle} g = f \upharpoonright \min\{\alpha \mid f(\alpha) \neq g(\alpha)\}$ .

For all  $\eta$ , let  $P_{\langle 1 \rangle \frown \eta}^M = P_{\eta}^N$ , and the ordering and the functions are naturally induced from  $N$ . The main point is that we set  $G_{\langle \rangle, \langle 1 \rangle}(f) = a_{\text{lg}(f)}$ . Now choose  $P_{\langle 0 \rangle \frown \eta}^M$ ,  $G_{\langle 0 \rangle \frown \eta, \langle 0 \rangle \frown \nu}$ , etc. arbitrarily, and let  $G_{\langle \rangle, \langle 0 \rangle}$  be any function.

Suppose that  $\langle f_i \mid i < \mu \rangle$  is a non-constant indiscernible sequence:

If  $f_1 < f_0$  (i.e.  $f_1 <_{\langle \rangle} f_0$ ), we shall have an infinite decreasing sequence in a well-ordered tree — a contradiction.

If  $f_0 < f_1$ ,  $\langle f_i \mid i < \mu \rangle$  is increasing, so  $\langle G_{\langle \rangle, \langle 1 \rangle}(f_i) = a_{\text{lg}(f_i)} \mid i < \mu \rangle$  is non-constant — contradiction (as it is an indiscernible sequence in  $M$  and hence in  $P_{\langle 0 \rangle}^N$ ).



Let  $h_i = f_0 \wedge f_{i+1}$  for  $i < \mu$  (where  $\wedge = \wedge_{\langle \rangle}$ ). This is an indiscernible sequence, and by the same arguments, it cannot increase or decrease, but as  $h_i \leq f_0$ , and  $(P_{\langle \rangle}, <_{\langle \rangle})$  is a tree, it follows that  $h_i$  is constant.

Assume  $f_0 \wedge f_1 < f_1 \wedge f_2$ , then  $f_{2i} \wedge f_{2i+1} < f_{2(i+1)} \wedge f_{2(i+1)+1}$  for all  $i < \mu$ , and again  $\langle f_{2i} \wedge f_{2i+1} \mid i < \mu \rangle$  an increasing indiscernible sequence and we have a contradiction.

By the same reasoning, it cannot be that  $f_0 \wedge f_1 > f_1 \wedge f_2$ . As  $(P_{\langle \rangle}, <_{\langle \rangle})$  is a tree, we conclude that  $f_0 \wedge f_2 = f_0 \wedge f_1 = f_1 \wedge f_2$ . But that is a contradiction (because if  $\alpha = \lg(f_0 \wedge f_1)$ , then  $|\{f_0(\alpha), f_1(\alpha), f_2(\alpha)\}| = 3$ ).

□

**2.3. In RCF there are no indiscernibles of  $\omega$ -tuples.** Here we show that a related phenomenon occurs in real closed fields. So here we assume  $\mathfrak{C} \models \text{RCF}$ .

*Notation 2.13.* The set of all open intervals  $(a, b)$  (where  $a < b$  and  $a, b \in \mathfrak{C}$ ) is denoted by  $\mathfrak{I}$ .

**Definition 2.14.** For a cardinal  $\kappa$ ,  $n \leq \omega$  and an ordinal  $\delta$ ,  $\kappa \rightarrow (\delta)_n^{\text{interval}}$  means: for every set  $A$  of (non-empty, open)  $n$ -tuples of intervals (so for each  $\bar{I} \in A$ ,  $\bar{I} = \langle I^i \mid i < n \rangle \in \mathfrak{I}^n$ ) of size  $\kappa$ , there is a sequence  $\langle \bar{I}_\alpha \mid \alpha < \delta \rangle \in {}^\delta A$  of order type  $\delta$  such that for  $\alpha < \beta < \delta$ ,  $\bar{I}_\alpha \neq \bar{I}_\beta$ , and  $\langle \bar{b}_\alpha \mid \alpha < \delta \rangle$  such that  $\bar{b}_\alpha \in \bar{I}_\alpha$  (i.e.  $\bar{b}_\alpha = \langle b_\alpha^0, \dots, b_\alpha^{n-1} \rangle$  and  $b_\alpha^i \in I_\alpha^i$ ) such that  $\langle \bar{b}_\alpha \mid \alpha < \delta \rangle$  is an indiscernible sequence.

*Remark 2.15.* Note that

- (1) If  $\kappa \rightarrow (\delta)_n^{\text{interval}}$  then  $\kappa \rightarrow (\delta)_m^{\text{interval}}$  for all  $m \leq n$ .
- (2) If  $\kappa \not\rightarrow (\delta)_n^{\text{interval}}$  then  $\kappa \not\rightarrow (\delta)_{\text{RCF}, n}$  (why? if  $A$  witness that  $\kappa \not\rightarrow (\delta)_n^{\text{interval}}$ , then for each  $\bar{I} \in A$ , choose  $\bar{b}_{\bar{I}} \in \bar{I}$  (as above) in such a way that  $\{\bar{b}_{\bar{I}} \mid \bar{I} \in A\}$  has size  $\kappa$ . By definition this set witnesses  $\kappa \not\rightarrow (\delta)_{\text{RCF}, n}$ ).
- (3) If  $\lambda < \kappa$  and  $\kappa \not\rightarrow (\delta)_n^{\text{interval}}$  then  $\lambda \not\rightarrow (\delta)_n^{\text{interval}}$ .

We shall prove the following theorem:

**Theorem 2.16.** *For any two cardinals  $\mu \leq \kappa$  such that in  $[\mu, \kappa]$  there are no strongly inaccessible cardinals,  $\kappa \not\rightarrow (\mu)_\omega^{\text{interval}}$ .*

The proof follows from a sequence of claims:

*Claim 2.17.* If  $\kappa < \mu$  then  $\kappa \not\rightarrow (\mu)_n^{\text{interval}}$  for all  $n \leq \omega$ .

*Proof.* Obvious. □

*Claim 2.18.* If  $\kappa = \mu = \aleph_0$  then  $\kappa \not\rightarrow (\mu)_1^{\text{interval}}$ .

*Proof.* For  $n < \omega$ , let  $I_n = (n, n+1)$ . □

*Claim 2.19.* Suppose  $\kappa = \sum_{i < \sigma} \lambda_i$  and  $n \leq \omega$ . Then, if  $\sigma \not\vdash (\mu)_n^{\text{interval}}$  and  $\lambda_i \not\vdash (\mu)_n^{\text{interval}}$  then  $\kappa \not\vdash (\mu)_{2+2n}^{\text{interval}}$ .

*Proof.* By assumption, we have a set of intervals  $\{\bar{R}_i \mid i < \sigma\}$  that witness  $\sigma \not\vdash (\mu)_n^{\text{interval}}$  and for each  $i < \sigma$  we have  $\{\bar{S}_\beta \mid \sum_{j < i} \lambda_j < \beta < \lambda_i\}$  that witness  $|\lambda_i \setminus \sum_{j < i} \lambda_j| \not\vdash (\mu)_n^{\text{interval}}$ .

Fix an increasing sequence of elements  $\langle b_i \mid i < \sigma \rangle$ .

For  $\alpha < \kappa$ , let  $\beta = \beta(\alpha) = \min\{i < \sigma \mid \alpha < \lambda_i\}$  and for  $i < 2 + 2n$ , define:

- If  $i = 0$ , let  $I_\alpha^i = (b_{2\beta(\alpha)}, b_{2\beta(\alpha)+1})$ .
- If  $i = 1$ , let  $I_\alpha^i = (b_{2\beta(\alpha)+1}, b_{2\beta(\alpha)+2})$ .
- If  $i = 2k + 2$ , let  $I_\alpha^i = R_{\beta(\alpha)}^k$ .
- If  $i = 2k + 3$ , let  $I_\alpha^i = S_\alpha^k$ .

Suppose  $\langle \bar{b}_\varepsilon \mid \varepsilon < \mu \rangle$  is an indiscernible sequence such that  $\bar{b}_\varepsilon \in \bar{I}_{\alpha_\varepsilon}$  for  $\varepsilon < \mu$ . Denote  $\bar{b}_\varepsilon = \langle b_0^\varepsilon, \dots, b_{2+2n-1}^\varepsilon \rangle$ . Note that  $b_1^\varepsilon < b_0^{\varepsilon'}$  if and only if  $\beta(\alpha_\varepsilon) < \beta(\alpha_{\varepsilon'})$  (we need two intervals for the “only if” direction).

Hence  $\langle \beta(\alpha_\varepsilon) \mid \varepsilon < \mu \rangle$  is increasing or constant. But if it is increasing then we have a contradiction to the choice of  $\{\bar{R}_i \mid i < \sigma\}$ . So it is constant, and suppose  $\beta(\alpha_\varepsilon) = i_0$  for all  $\varepsilon < \mu$ . But then  $\alpha_\varepsilon \in \lambda_{i_0} \setminus \sum_{j < i_0} \lambda_j$  for all  $\varepsilon < \mu$  and we get a contradiction to the choice of  $\langle \bar{S}_\beta \mid \sum_{j < i_0} \lambda_j < \beta < \lambda_{i_0} \rangle$ .  $\square$

*Claim 2.20.* Suppose  $\lambda \not\vdash (\mu)_n^{\text{interval}}$ . Then  $2^\lambda \not\vdash (\mu)_{4+2n}^{\text{interval}}$ .

*Proof.* Suppose  $\{\bar{I}_\alpha \mid \alpha \leq \lambda\}$  witnesses that  $\lambda \not\vdash (\mu)_n^{\text{interval}}$ .

By adding two intervals to each  $\bar{I}_\alpha$ , we can ensure that it has the extra property that if  $\bar{c}_1 \in I_{\alpha_1}$  and  $\bar{c}_2 \in \bar{I}_{\alpha_2}$  then  $c_1^1 < c_2^0$  if and only if  $\alpha_1 < \alpha_2$  (as in the previous claim). By this we have increased the length of  $\bar{I}_\alpha$  to  $2 + n$  (and it is still a witness of  $\lambda \not\vdash (\mu)_n^{\text{interval}}$ ).

We write  $\bar{c}_1 <^* \bar{c}_2$  for  $c_1^1 < c_2^0$ , but note that it is not really an ordering (it is not transitive).

We shall find below a four-place definable function  $f$  such that:

- $\heartsuit$  For every two ordinals,  $\delta, \zeta$ , if  $\langle \bar{R}_\alpha \mid \alpha < \delta \rangle$  is a sequence of  $\zeta$ -tuples of intervals, then there exists a set of  $2\zeta$ -tuples of intervals,  $\{\bar{S}_\eta \mid \eta \in {}^\delta 2\}$  such that for all  $i < \zeta$  and  $\eta_1 \neq \eta_2$ , if  $b_1 \in S_{\eta_1}^{2i}, b_2 \in S_{\eta_1}^{2i+1}$  and  $b_3 \in S_{\eta_2}^{2i}, b_4 \in S_{\eta_2}^{2i+1}$  then  $f(b_1, b_2, b_3, b_4)$  is in  $R_{\text{lg}(\eta_1 \wedge \eta_2)}^i$ .

Apply  $\heartsuit$  to our situation to get  $\{\bar{J}_\eta \mid \eta \in {}^\lambda 2\}$  such that  $\bar{J}_\eta = \langle J_\eta^i \mid i < 4 + 2n \rangle$  and for all  $k < 2 + n$  and  $\eta_1 \neq \eta_2$ , if  $b_1 \in J_{\eta_1}^{2k}, b_2 \in J_{\eta_1}^{2k+1}$  and  $b_3 \in J_{\eta_2}^{2k}, b_4 \in J_{\eta_2}^{2k+1}$  then  $f(b_1, b_2, b_3, b_4)$  is in  $I_{\text{lg}(\eta_1 \wedge \eta_2)}^k$ .

This is enough (the reasons are exactly as in the regular case of the proof of Theorem 2.11, but we shall repeat it for clarity):

To simplify notation, we regard  $f$  as a function on tuples, so that if  $\bar{b}_1 \in \bar{J}_{\eta_1}, \bar{b}_2 \in \bar{J}_{\eta_2}$  then  $f(\bar{b}_1, \bar{b}_2)$  is in  $\bar{I}_{\text{lg}(\eta_1 \wedge \eta_2)}$  (namely,  $f(\bar{b}_1, \bar{b}_2) = \langle a_k \mid k < 2 + n \rangle$  where  $a_k = f(b_{2k}^1, b_{2k+1}^1, b_{2k}^2, b_{2k+1}^2) \in I_{\text{lg}(\eta_1 \wedge \eta_2)}^k$  for  $k < 2 + n$ ).

Suppose  $\langle \eta_i \mid i < \mu \rangle \subseteq {}^\lambda 2$  is without repetitions and  $\langle \bar{b}_{\eta_i} \mid i < \mu \rangle$  is an indiscernible sequence such that  $\bar{b}_{\eta_i} \in \bar{J}_{\eta_i}$ .

Let  $h_i = \eta_0 \wedge \eta_{i+1}$  for  $i < \mu$ . If  $\lg(h_i) < \lg(h_j)$  for some  $i \neq j$  then  $f(\bar{b}_{\eta_0}, \bar{b}_{\eta_{i+1}}) <^* f(\bar{b}_{\eta_0}, \bar{b}_{\eta_{j+1}})$  and so by indiscernibility,  $\langle \lg(h_i) \mid i < \mu \rangle$  is increasing (it cannot be decreasing), and so  $f(\bar{b}_{\eta_0}, \bar{b}_{\eta_{i+1}})$  contradicts our choice of  $\langle \bar{I}_\alpha \mid \alpha < \lambda \rangle$ . Hence (because  $h_i \leq \eta_0$ )  $h_i$  is constant.

Assume  $\eta_0 \wedge \eta_1 < \eta_1 \wedge \eta_2$ , then  $f(\bar{b}_{\eta_0}, \bar{b}_{\eta_1}) <^* f(\bar{b}_{\eta_1}, \bar{b}_{\eta_2})$  so  $f(\bar{b}_{\eta_1}, \bar{b}_{\eta_2}) <^* f(\bar{b}_{\eta_2}, \bar{b}_{\eta_3})$ , and so  $\lg(\eta_0 \wedge \eta_1) < \lg(\eta_1 \wedge \eta_2) < \lg(\eta_2 \wedge \eta_3)$  hence,  $f(\bar{b}_{\eta_0}, \bar{b}_{\eta_1}) <^* f(\bar{b}_{\eta_2}, \bar{b}_{\eta_3})$  and it follows that  $\langle \lg(\eta_{2i} \wedge \eta_{2i+1}) \mid i < \mu \rangle$  is increasing. And this is again a contradiction.

Similarly, it cannot be that  $\eta_0 \wedge \eta_1 > \eta_1 \wedge \eta_2$ . As both sides are less or equal than  $\eta_1$ , it must be that  $\eta_0 \wedge \eta_2 = \eta_0 \wedge \eta_1 = \eta_1 \wedge \eta_2$ . But that is impossible (because if  $\alpha = \lg(\eta_0 \wedge \eta_1)$ , then  $|\{\eta_0(\alpha), \eta_1(\alpha), \eta_2(\alpha)\}| = 3$ ).

*Claim.*  $\heartsuit$  is true.

*Proof.* Let  $f(x, y, z, w) = (x - z) / (y - w)$  (do not worry about division by 0, we shall explain below).

It is enough, by the definition of  $\heartsuit$ , to assume  $\zeta = 1$ . By compactness, we may assume that  $\delta$  is finite, and to avoid confusion, denote it by  $m$ . So we have a finite tree,  ${}^m 2$ , and a sequence of intervals  $\langle R_i \mid i < m \rangle$ . Each  $R_i$  is of the form  $(a_i, b_i)$ . Let  $c_i = (b_i + a_i) / 2$ . Let  $d \in \mathfrak{C}$  be any element greater than any member of  $A := \text{acl}(a_i, b_i \mid i < m)$ . For each  $\eta \in {}^m 2$ , let  $a_\eta = \sum_{i < m} \eta(i) c_i d^{m-i}$ , and  $b_\eta = \sum_{i < m} \eta(i) d^{m-i}$ .

Let  $S_\eta^0 = (a_\eta - 1, a_\eta + 1)$  and  $S_\eta^1 = (b_\eta - 1, b_\eta + 1)$ .

This works:

Assume  $b_1 \in S_{\eta_1}^0$ ,  $b_2 \in S_{\eta_1}^1$  and  $b_3 \in S_{\eta_2}^0$ ,  $b_4 \in S_{\eta_2}^1$ .

We have to show  $(b_1 - b_3) / (b_2 - b_4) \in R_{\lg(\eta_1 \wedge \eta_2)}$ . Denote  $k = \lg(\eta_1 \wedge \eta_2)$ .

$a_{\eta_1} - a_{\eta_2}$  is of the form  $\varepsilon c_k d^{m-k} + F(d)$  where  $\varepsilon \in \{-1, 1\}$ , and  $F(d)$  is a polynomial over  $A$  of degree  $\leq m - k - 1$ .  $b_{\eta_1} - b_{\eta_2}$  is of the form  $\varepsilon d^{m-k} + G(d)$ , where  $\varepsilon$  is the same for both (and  $G$  is a polynomial over  $\mathbb{Z}$  of degree  $\leq m - k - 1$ ). Now,  $b_1 - b_3 \in (a_{\eta_1} - a_{\eta_2} - 2, a_{\eta_1} - a_{\eta_2} + 2)$ , and  $b_2 - b_4 \in (b_{\eta_1} - b_{\eta_2} - 2, b_{\eta_1} - b_{\eta_2} + 2)$ , and hence we know that  $b_2 - b_4 \neq 0$ . It follows that  $(b_1 - b_3) / (b_2 - b_4)$  is inside an interval whose endpoints are  $\{(\varepsilon c_k d^{m-k} + F(d) \pm 2) / (\varepsilon d^{m-k} + G(d) \pm 2)\}$ . But

$$(\varepsilon c_k d^{m-k} + F(d) \pm 2) / (\varepsilon d^{m-k} + G(d) \pm 2) \in R_k$$

by our choice of  $d$ , and we are done. □

□

The proof of Theorem 2.16 now follows by induction on  $\kappa$ : Fix  $\mu$ , and let  $\kappa$  be the first cardinal for which the theorem fails. Then by Claim 2.17,  $\kappa \geq \mu$ . By Claim 2.18,  $\aleph_0 < \kappa$ . By Claim 2.19,

$\kappa$  cannot be singular. By Claim 2.20,  $\kappa$  cannot be regular, because if it were, there would be a  $\lambda < \kappa$  such that  $2^\lambda \geq \kappa$  (because  $\kappa$  is not strongly inaccessible). Note that we did use claim 2.17 to deal with cases where we couldn't use the induction hypothesis (for example, in the regular case, it might be that  $\lambda < \mu$ ).

*Further remarks.* Theorem 2.16 can be generalized to allow parameters:

Suppose  $\mathfrak{C} \models \text{RCF}$ , and  $A \subseteq \mathfrak{C}$ .

**Definition 2.21.**  $\kappa \rightarrow_A (\mu)_\omega^{\text{interval}}$  means the same as in definition 2.14, but we require that the indiscernible sequence is indiscernible over  $A$ .

Then we have:

**Theorem 2.22.** *For any set of parameters  $A$  such that in  $[|A|, \kappa]$  there are no strongly inaccessible cardinals or  $\kappa \leq |A|$ ,  $\kappa \not\rightarrow_A (\aleph_0)_\omega^{\text{interval}}$ .*

*Proof.* The proof goes exactly as the proof of Theorem 2.16, but the base case for the induction is different. There it was  $\mu = \kappa = \aleph_0$ . Here it is  $\kappa \leq |A|$ . Indeed, enumerate  $A = \{a_i \mid i < \mu\}$ . Let  $\varepsilon \in \mathfrak{C}$  be greater than 0 but smaller than any element in  $\text{acl}(A)$ . For  $i < \mu$ , let  $I_i = (a_i, a_i + \varepsilon)$ . Then  $\{I_i \mid i < \kappa\}$  witnesses  $\kappa \not\rightarrow_A (\aleph_0)_\omega^{\text{interval}}$ .  $\square$

### 3. GENERIC PAIR

Here we give an example of an  $\omega$ -stable theory, such that for all weakly generic pairs of structures  $M \prec M_1$  the theory of the pair  $(M_1, M)$  in an extended language where we name  $M$  by a predicate has the independence property.

**Definition 3.1.** A pair  $(M_1, M)$  as above is *weakly generic* if for all formula  $\varphi(x)$  with parameters from  $M$ , if  $\varphi$  has infinitely many solutions in  $M$ , then it has a solution in  $M_1 \setminus M$ .

This definition is induced by the well known “generic pair conjecture” (see [She, She12]), and it is worth while to give the precise definitions.

**Definition 3.2.** Assume  $\lambda = \lambda^{<\lambda} > |T|$  (in particular,  $\lambda$  is regular),  $2^\lambda = \lambda^+$ , and for all  $\alpha < \lambda^+$ ,  $M_\alpha \models T$  is of cardinality  $\lambda$ . Suppose  $\langle M_\alpha \mid \alpha < \lambda^+ \rangle$  is an increasing continuous sequence. Furthermore,  $M = \bigcup_{\alpha < \lambda^+} M_\alpha$  is a saturated model of cardinality  $\lambda^+$ . The generic pair property says that there exists a club  $E \subseteq \lambda^+$  such that for all  $\alpha < \beta \in E$  of cofinality  $\lambda$ , the pair  $(M_\beta, M_\alpha)$  has the same isomorphism type. We call this pair the *generic* pair of  $T$  of size  $\lambda$ .

**Proposition 3.3.** *The generic pair property and the generic pair depend only on  $T$  and  $\lambda$ , and not on the particular choice of  $M$  or  $\langle M_\alpha \rangle$ .*

*Proof.* Suppose  $M$  satisfies Definition 3.2. This means that  $|M| = \lambda^+$ ,  $M$  is saturated, there are  $\langle M_\alpha \mid \alpha < \lambda^+ \rangle$  such that  $M = \bigcup_{\alpha < \lambda^+} M_\alpha$ , and there is a club  $E \subseteq \lambda^+$  such that for all  $\alpha, \beta \in E$  such that  $\alpha < \beta$ , the pair  $(M_\beta, M_\alpha)$  has the same isomorphism type. If  $N$  is another saturated model of size  $\lambda^+$  and  $N = \bigcup_{\alpha < \lambda^+} N_\alpha$ . Then  $N \cong M$ , so we may assume  $N = M$ . Let  $E_0 = \{\delta < \lambda^+ \mid N_\delta = M_\delta\}$ . This is a club of  $\lambda^+$ , and so  $E \cap E_0$  is also a club of  $\lambda^+$  such that  $(N_\beta, N_\alpha)$  has the same isomorphism type for any  $\alpha < \beta \in E \cap E_0$  of cofinality  $\lambda$ .  $\square$

Justifying definition 3.1 we have:

*Claim 3.4.* Assume that  $T$  has the generic pair property for  $\lambda$ , then every generic pair is weakly generic.

*Proof.* So suppose that  $E, \langle M_\alpha \mid \alpha < \lambda^+ \rangle$  are as in Definition 3.2. Suppose  $\alpha, \beta \in E$  and  $\alpha < \beta$ . We are given a formula  $\varphi(x)$  with parameter from  $M_\alpha$ , such that  $\aleph_0 \leq |\varphi(M_\alpha)|$ . But by saturation of  $M$ ,  $\lambda^+ = |\varphi(M)|$ , and  $M = \bigcup_{\alpha \in E} M_\alpha$ , hence there is some  $\alpha < \beta' \in E$  such that  $\varphi(M_\beta) \neq \emptyset$ , but as  $(M_\beta, M_\alpha) \cong (M_{\beta'}, M_\alpha)$ , we are done.  $\square$

The generic pair property is very important in dependent theories. In fact, they characterize them: In [She06, She12, She, She11] it is proved that  $T$  is dependent if and only if for large enough  $\lambda = \lambda^{<\lambda}$ ,  $T$  has the generic pair property.

Hence it makes sense to ask whether the theory of the pair is dependent.

The answer is no:

**Theorem 3.5.** *There exists an  $\omega$ -stable theory such that for every weakly generic pair of models  $M \prec M_1$ , the theory of the pair  $(M_1, M)$  has the independence property.*

We shall describe this theory:

Let  $L = \{P, R, Q_1, Q_2\}$  where  $R, P$  are unary predicates and  $Q_1, Q_2$  are binary relations.

Let  $\tilde{M}$  be the following structure for  $L$ :

(1) The universe is

$$\begin{aligned} \tilde{M} &= \{u \subseteq \omega \mid |u| < \omega\} \cup \\ &\quad \{(u, v, i) \mid u, v \subseteq \omega, |u| < \omega, |v| < \omega, i < \omega \ \& \ u \subseteq v \Rightarrow i < |v| + 1\}. \end{aligned}$$

(2) The predicates are interpreted as follows:

- $P^{\tilde{M}} = \{u \subseteq \omega \mid |u| < \aleph_0\}$ .
- $R^{\tilde{M}}$  is  $M \setminus (P^{\tilde{M}})$ .
- $Q_1^{\tilde{M}} = \{(u, (u, v, i)) \mid u \in P^{\tilde{M}}\}$ .
- $Q_2^{\tilde{M}} = \{(v, (u, v, i)) \mid v \in P^{\tilde{M}}\}$ .

Let  $T = \text{Th}(\tilde{M})$ .

As we shall see in the next claim,  $T$  gives rise to the following definition:

**Definition 3.6.** We call a structure  $(B, \cup, \cap, -, \subseteq, 0)$  a *pseudo Boolean algebra* (PBA) when it satisfies all the axioms of a Boolean algebra except: There is no greatest element  $1$  (i.e. remove all the axioms concerning it).

Pseudo Boolean algebra can have atoms like in Boolean algebras (nonzero elements that do not contain any smaller nonzero elements).

**Definition 3.7.** Say that a PBA is of *finite type* if every element is a union of finitely many atoms.

**Definition 3.8.** For a PBA  $A$ , and  $C \subseteq A$  a sub-PBA, let  $A_C := \{a \in A \mid \exists c \in C (a \subseteq^A c)\}$ , and for a subset  $D \subseteq A$ , let  $\text{at}(D)$  be the set of atoms contained in  $D$ .

**Proposition 3.9.** *Every PBA of finite type is isomorphic to  $(\mathcal{P}_{<\infty}(\kappa), \cup, \cap, -, \subseteq, \emptyset)$  for some  $\kappa$  where  $\mathcal{P}_{<\infty}(\kappa)$  is the set of all finite subsets of  $\kappa$ . Moreover: Assume  $A, B$  are PBAs of finite type and  $C \subseteq A, B$  is a common sub-PBA. Then, if*

- (1)  $|\text{at}(A \setminus A_C)| = |\text{at}(B \setminus B_C)|$ .
- (2) *For every  $c \in C$ ,  $A$  and  $B$  agree on the size of  $c$  (the number of atoms it contains).*

*Then there is an isomorphism of PBAs  $f : A \rightarrow B$  such that  $f \upharpoonright C = \text{id}$ .*

*Proof.* The first part follows from the easy observation that in a PBA of finite type, every element has a unique presentation as a union of finitely many atoms. So if  $A$  is a PBA, and its set of atoms is  $\{a_i \mid i < \kappa\}$ , then take  $a_i$  to  $\{i\}$ .

For the moreover part, first we extend  $\text{id}_C$  to an isomorphism from  $A_C$  to  $B_C$ : consider all elements in  $C$  of minimal size, these are the atoms of  $C$ . For each such  $c$ , map the set of atoms in  $A$  contained in  $c$  to the set of atoms in  $B$  contained in  $c$ . This is well defined and can be extended this to all of  $A_C$ .

Now,  $|\text{at}(A \setminus A_C)| = |\text{at}(B \setminus B_C)|$ , so any bijection between the set of atoms induces an isomorphism.  $\square$

*Claim 3.10.*  $T$  is  $\omega$ -stable.

*Proof.* We prove that an extension of  $T$  to a larger vocabulary is  $\omega$ -stable, by adding new relations to the language, which are all definable —

$$\{S_n, \subseteq_n, \pi_1, \pi_2, \cap_n, \cup_n, -, e \mid n \geq 1\}$$

where  $S_n$  is a unary relation defined on  $P$ ,  $\subseteq_n$  is a binary relation defined on  $P$ ,  $\pi_1, \pi_2$  are two unary functions from  $R$  to  $P$ ,  $\cap_n, -_n$  are binary functions from  $S_n$  to  $S_n$  and  $e$  is a constant in  $P$ . Their interpretation in  $\tilde{M}$  are as follows:

- $\pi_1((u, v, i)) = u, \pi_2((u, v, i)) = v.$
- For each  $1 \leq n < \omega$ ,  $S_n(v) \Leftrightarrow |v| \leq n.$
- For each  $1 \leq n$ ,  $u \subseteq_n v$  if and only if  $|u| \leq n, |v| \leq n$  and  $u \subseteq v.$
- $u \cap_n v = u \cap v$  for all  $u, v \in S_n.$
- $u -_n v = u \setminus v$  for  $v \subseteq_n u.$
- $u \cup_n v = u \cup v$  for  $u, v \in S_n.$
- $e = \emptyset.$

Note that they are indeed definable:

- (1)  $\pi_1(x)$  is the unique  $y$  such that  $Q_1(y, x)$ , and similarly  $\pi_2$  is definable.
- (2) Let  $E(x, y)$  by an auxiliary equivalence relation defined by  $\pi_1(x) = \pi_1(y) \wedge \pi_2(x) = \pi_2(y).$
- (3)  $e$  is the unique element  $x \in P$  such that there exists exactly one element  $z \in R$  such that  $\pi_1(z) = x = \pi_2(z).$
- (4)  $x \subseteq_n y$  is defined by “ $P(x), P(y)$  and the number of elements in the  $E$  class of some (equivalently any) element  $z$  such that  $\pi_1(z) = x, \pi_2(z) = y$  is at most  $n + 1$ ”.
- (5)  $S_n(x)$  is defined by “ $P(x)$  and  $e \subseteq_n x$ ” (In particular,  $e \in S_n$  for all  $n$ ).
- (6)  $\cap_n$  and  $-_n$  are then naturally definable using  $\subseteq_n$ . For instance  $x -_n y = z$  if and only if  $x, y, z$  are in  $S_n$ ,  $z \subseteq_n x$  and for each  $e \neq w \subseteq_n y$ ,  $w \not\subseteq_n z.$
- (7)  $x \cup_n y = z$  if and only if  $x, y \in S_n, z \in S_{2n}$ ,  $x, y \subseteq_{2n} z$  and  $z -_{2n} x \subseteq_{2n} y.$

Furthermore,  $\subseteq_k \upharpoonright S_n = \subseteq_n$  for  $n \leq k$ . Hence every model  $M$  of  $T$  gives rise naturally to an induced PBA:  $B^M := (\bigcup_n S_n^M, \cup^M, \cap^M, -^M, \subseteq^M, e^M)$  where  $\cup^M = \bigcup \{\cup_n^M \mid n < \omega\}$ , and similarly for  $\subseteq^M, -^M$  and  $\cap^M$  (see Definition 3.6 above).

*Claim.* In the extended language,  $T$  eliminates quantifiers.

*Proof.* Suppose  $M, N \models T$  are saturated models,  $|M| = |N|$  and  $A \subseteq M, N$  is a common substructure (where  $|A| < |M|$ ). It is enough to show that we have an isomorphism from  $M$  to  $N$  fixing  $A$ .

By Proposition 3.9, we have an isomorphism  $f$  from  $B^M$  to  $B^N$  preserving  $A$  (by saturation and the choice of language, the condition of the proposition are satisfied).

On  $P^M \setminus (B^M \cup A)$  there is no structure and it has the same size as  $P^N \setminus (B^N \cup A)$  (namely  $|N|$ ), so we can extend the isomorphism  $f$  to  $P^M$ .

We are left with  $R^M$ : let  $a \in R^M$ , and  $a_i = \pi_i(a)$  for  $i = 1, 2$ . We already know  $f(a_1), f(a_2)$ . Suppose  $a_1 \subseteq_n a_2$  for minimal  $n$ . Then there are exactly  $n$  elements  $z \in R^M$  with  $\pi_1(z) =$

$\alpha_1, \pi_2(z) = \alpha_2$ . This is true also in  $\mathcal{R}^N$ , and the number of such  $z$ 's not in  $A$  is the same for both  $M, N$ . Hence we can take this E-equivalence class from  $M$  to the appropriate class in  $N$ .

If not, i.e.  $\alpha_1 \not\subseteq_n \alpha_2$  for all  $n$ , then there are infinitely many elements  $z$  in  $N$  and in  $M$  with  $\pi_1(z) = \alpha_1, \pi_2(z) = \alpha_2$ , and again we take this E-class in  $M$  outside of  $A$  to the appropriate E-class in  $N$ .  $\square$

Now we can conclude the proof by a counting types argument. Let  $M$  be a countable model of  $T$ . Let  $p(x)$  be a non-algebraic type over  $M$ . There are some cases:

*Case 1.*  $S_n(x) \in p$  for some  $n$ . Then the type is determined by the maximal element  $c$  in  $M$  such that  $c \subseteq_n x$  (this is easy, but also follows from Proposition 3.9).

*Case 2.*  $S_n(x) \notin p$  for all  $n$  but  $P(x) \in p$ . Then  $x$  is already determined — there is nothing more we can say on  $x$ .

*Case 3.*  $R(x) \in p$ . Then the type of  $x$  is determined by the type of  $(\pi_1(x), \pi_2(x))$  over  $M$ .

So the number of types over  $M$  is countable.  $\square$

**Proposition 3.11.** *Every weakly generic pair of models of  $T$  has the independence property.*

*Proof.* Suppose  $(M_1, M)$  is a weakly generic pair. We think of it as a structure of the language  $L_Q$ , where  $Q$  is interpreted as  $M$ . Consider the formula

$$\varphi(x, y) = P(x) \wedge P(y) \wedge \exists z \notin Q(Q_1(x, z) \wedge Q_2(y, z)).$$

This formula has IP: Let  $\{\alpha_i \mid i < \omega\} \subseteq M$  be elements from  $P^M$  such that  $\alpha \in S_1^M$  (as in the language of the proof of Claim 3.10), i.e. they are atoms in the induced PBA, and  $\alpha_i \neq \alpha_j$  for  $i \neq j$ . For any finite  $s \subseteq \omega$  of size  $n$ , there is an element  $b_s \in P^M$  be such that  $\alpha_i \subseteq_n^M b_s$  for all  $i \in s$ . Then for all  $i \in \omega$ ,  $\varphi(\alpha_i, b_s)$  if and only if  $i \notin s$ :

If  $\varphi(\alpha_i, b_s)$  there are infinitely many  $z$ 's in  $M$  such that  $Q_1(\alpha_i, z) \wedge Q_2(b_s, z)$  (otherwise they would all be in  $M$ ). This means that  $\alpha_i \not\subseteq_n^M b_s$  so  $i \notin s$ .

For the other direction, the same exact argument works, but this time use the fact that the pair is weakly generic.  $\square$

## 4. DIRECTIONALITY

### 4.1. Introduction to Directionality.

**Definition 4.1.** A global type  $p(x) \in S(\mathcal{C})$  is said to be *finitely satisfiable* over a set  $A$ , or a *co-heir* over  $A$  if for every formula  $\varphi(x, y)$ , if  $\varphi(x, b) \in p$ , then for some  $a \in A$ ,  $\varphi(a, b)$  holds.



It is well known (see [Adl08]) that a theory  $T$  is dependent if and only if given a type  $p(x) \in S(M)$  over a model  $M$ , the number of complete global types  $q \in S(\mathfrak{C})$  that extend  $p$  and are finitely satisfiable in  $M$  is at most  $2^{|M|}$  (while the maximal number is  $2^{2^{|M|}}$ ).

Here we analyze the behavior of the number of global co-heir extensions in a dependent theory. We classify theories by what we call directionality:

The first possibility is that the number of such global extensions is small, i.e. that the number does not depend on the model but only on the theory (theories with *small* directionality); the second possibility is that this number is bounded by  $|M|^{|T|}$  (*medium* directionality); and the third one is that this number is greater than  $|M|^{|T|}$  (*large* directionality).

As far as we know, the first to give an example of a dependent theory with large directionality was Delon in [Del84].

Here we give a simple combinatorial examples using the constructions of Section 2 for each of the possible directionalities, and furthermore we show that RCF is large.

We also give a local characterization of directionality. Namely, for a finite set of formulas  $\Delta$ , we show that  $T$  has small directionality if and only if the number of  $\Delta$ -co-heirs that extend a type  $p \in S(M)$  is finite.  $T$  has medium directionality if this number is  $|M|$ , and it has large directionality if it is at least  $\text{ded}|M|$ .

We do not always assume that  $T$  is dependent in this section.

**Definition 4.2.** For a type  $p \in S(A)$ , let:

$$\text{uf}(p) = \{q \in S(\mathfrak{C}) \mid q \text{ is a co-heir extension of } p \text{ over } A\}.$$

For a partial type  $p(x)$  over a set  $A$ , and a set of formulas  $\Delta$ ,

$$\text{uf}_\Delta(p) = \{q(x) \in S_\Delta(\mathfrak{C}) \mid q \cup p \text{ is f.s. in } A\}.$$

Note: this definition only makes sense if  $p$  is finitely satisfiable in  $A$ . The notation  $\text{uf}$  refers to ultra-filter.

And here is the main definition of this section:

**Definition 4.3.** Let  $T$  be any theory, then:

- (1)  $T$  is said to have *small* directionality (or just,  $T$  is small) if and only if for all finite  $\Delta$ ,  $M \models T$  and  $p \in S(M)$ ,  $\text{uf}_\Delta(p)$  is finite.
- (2)  $T$  is said to have *medium* directionality (or just,  $T$  is medium) if and only if for every  $\lambda \geq |T|$ ,

$$\lambda = \sup\{|\text{uf}_\Delta(p)| \mid p \in S(M), \Delta \text{ finite}, |M| = \lambda\}.$$

- (3)  $T$  is said to have *large* directionality (or just,  $T$  is large) if  $T$  is neither small nor medium.

**Observation 4.4.** *If  $T$  has the independence property, then it is large. In fact, if  $\varphi(x, y)$  has the independence property, then there is a type  $p(x)$  over a model  $M$ , that has  $2^{2^{|M|}}$  many  $\{\varphi(x, y)\}$ -extensions that are finitely satisfiable in  $M$ .*

*Proof.* We may assume that  $T$  has Skolem functions. Let  $\lambda \geq |T|$ , and let  $\bar{a} = \langle a_i \mid i < \lambda \rangle$ ,  $\langle b_s \mid s \subseteq \lambda \rangle$  be such that  $\langle a_i \mid i < \lambda \rangle$  is indiscernible and  $\varphi(a_i, b_s)$  holds iff  $i \in s$ . Let  $M$  be the Skolem hull of  $\bar{a}$ . Let  $p(x) \in S(M)$  be the limit of  $\bar{a}$  in  $M$  (so  $\psi(x, c) \in p$  iff  $\psi(x, c)$  holds for an end segment of  $\bar{a}$ ). Let  $P \subseteq \mathcal{P}(\lambda)$  be an independent family of size  $2^\lambda$  (i.e. such that every finite Boolean combination has size  $\lambda$ ). Then for each  $D \subseteq P$ ,  $p(x) \cup \left\{ \varphi(x, b_s)^{s \in D} \mid s \subseteq \lambda \right\}$  is finitely satisfiable in  $M$ .  $\square$

4.1.1. *Small directionality.* The following construction will be useful (here and in the Section 5):

**Construction 4.5.** *Suppose that there is some  $p \in S(M)$  and finite  $\Delta$  such that  $\text{uf}_\Delta(p)$  is infinite, and contains  $\{q_i \mid i < \omega\}$ .*

*For all  $i < j < \omega$ , there is a formula  $\varphi_{i,j} \in \Delta$  and  $b_{i,j} \in \mathfrak{C}$  such that  $\varphi_{i,j}(x, b_{i,j}) \in q_i$ ,  $\neg \varphi_{i,j}(x, b_{i,j}) \in q_j$  (or the other way around). By Ramsey's Theorem we may assume  $\varphi_{i,j}$  is constant —  $\varphi(x, y)$ . Let  $N$  be a model containing  $M$  and  $\{b_{i,j} \mid i < j < \omega\}$ .*

*Suppose  $\langle c_i \mid i < \omega \rangle$  are in  $\mathfrak{C}$  and  $c_i \models q_i|_N$ . Let  $N'$  be a model containing  $\{c_i \mid i < \omega\} \cup N$ . Let  $M^* = (N', N, M, Q, \bar{f})$  where  $Q = \{c_i \mid i < \omega\}$  and  $\bar{f} : Q^2 \rightarrow N$  is a tuple of functions of length  $\lg(y)$  defined by  $\bar{f}(c_i, c_j) = \bar{f}(c_j, c_i) = b_{i,j}$  for  $i < j$ .*

*So if  $N \models \text{Th}(M^*)$  then  $N = (N'_0, N_0, M_0, Q_0, \bar{f}_0)$  and*

- $M_0 \prec N_0 \prec N'_0 \models T$ ,
- $N_0 \cup Q_0 \subseteq N'_0$ ,
- $\bar{f}_0$  are functions from  $Q_0^2$  to  $N_0$ ,
- For all  $c, d \in Q_0$ ,  $c \equiv_{M_0} d$ ,
- $\text{tp}_\Delta(c/N_0)$  is finitely satisfiable in  $M_0$  for all  $c \in Q_0$ , and
- $\varphi(c, \bar{f}_0(c, d)) \Delta \varphi(d, \bar{f}_0(c, d))$  (where  $\Delta$  denotes symmetric difference) holds for all  $c \neq d \in Q_0$ .

*Claim 4.6.* Let  $T$  be any theory. Then  $T$  is small if and only if for every type  $p(x) \in S(M)$ ,  $|\text{uf}(p)| \leq 2^{|T|}$  (here  $p$  can also be an infinitary type, but then the bound is  $2^{|T| + |\lg(x)|}$ ).

In addition, if  $T$  is not small, then for every  $\lambda \geq |T|$ , there is a model  $M \models T$  of cardinality  $\lambda$ , a type  $p \in S(M)$ , and a finite set of formulas  $\Delta$  such that  $|\text{uf}_\Delta(p)| \geq \lambda$ .

*Proof.* Assume that  $T$  is small. The injective function  $\text{uf}(p) \rightarrow \prod_{\varphi \in L} \text{uf}_{\{\varphi\}}(p)$  shows that  $|\text{uf}(p)| \leq 2^{|T|}$ .

Conversely (and the “In addition” part): Assume that there is some  $p$  and  $\Delta$  such that  $\text{uf}_\Delta(p)$  is infinite. Use Construction 4.5: For every  $\lambda \geq |T|$  we may find  $N \models \text{Th}(M^*)$  of size  $\lambda$  such that

$|M_0| = |Q_0| = \lambda$ , and we have a model  $M_0$  of  $T$  with a type  $p$  over it, which have at least  $\lambda$  many  $\Delta$ -co-heirs of  $p$ .  $\square$

We conclude this section with a claim on theories with non-small or large directionality.

*Claim 4.7.* Suppose  $T$  has medium or large directionality. Then there exists some  $M \models T$ ,  $p \in S(M)$ ,  $\psi(x, y)$  and  $\{c_i \mid i < \omega\} \subseteq \mathfrak{C}$  such that for each  $i < \omega$  the set  $p(x) \cup \{\psi(x, c_j)^{j=i} \mid j < \omega\}$  is finitely satisfiable in  $M$ .

*Proof.* We consider the structure  $M^*$  introduced in Construction 4.5 and the formula  $\varphi$  chosen there. Find an extension of with an indiscernible sequence  $\langle d_i \mid i \in \mathbb{Z} \rangle$  inside  $Q$ . Assume without loss that  $\varphi(d_0, \bar{f}(d_0, d_1)) \wedge \neg \varphi(d_1, \bar{f}(d_0, d_1))$  holds. This means that  $\varphi(d_0, \bar{f}(d_0, d_1)) \wedge \neg \varphi(d_0, \bar{f}(d_{-1}, d_0))$ .

We claim that  $\varphi(d_i, \bar{f}(d_j, d_{j+1})) \wedge \neg \varphi(d_i, \bar{f}(d_{j-1}, d_j))$  holds if and only if  $i = j$ :

Suppose this holds but  $i \neq j$ . If  $i > j$  then, since  $\neg \varphi(d_1, \bar{f}(d_0, d_1))$ , it must be that  $i > j + 1$ , but then we have a contradiction to indiscernibility. Similarly, it cannot be that  $i < j$ . Thus the claim is proved with  $\psi(x; y, z) = \varphi(x, y) \wedge \neg \varphi(x, z)$ , and  $c_i = \langle f(d_i, d_{i+1}), f(d_{i-1}, d_i) \rangle$ .  $\square$

4.1.2. *Some helpful facts about dependent theories.* Assume  $T$  is dependent.

Recall,

**Definition 4.8.** A global type  $p(x)$  is *invariant over* a set  $A$  if it does not split over it, namely if whenever  $b$  and  $c$  have the same type over  $A$ ,  $\varphi(x, b) \in p$  if and only if  $\varphi(x, c) \in p$  for every formula  $\varphi(x, y)$ .

**Definition 4.9.** Suppose  $p(x)$  and  $q(y)$  are global  $A$ -invariant types. Then  $(q \otimes p)(x, y)$  is a global invariant type defined as follows: for any  $B \supseteq A$ , let  $a_B \models p|_B$  and  $b_B \models q|_{B \setminus A}$ , then  $p \otimes q = \bigcup_{B \supseteq A} \text{tp}(a_B, b_B/B)$ . One can easily check that it is well defined and  $A$ -invariant. Let  $p^{(n)} = p \otimes p \dots \otimes p$  where the product is done  $n$  times. So  $p^{(n)}$  is a type in  $(x_0, \dots, x_{n-1})$ , and  $p^{(\omega)} = \bigcup_{n < \omega} p^{(n)}$  is a type in  $(x_0, \dots, x_n, \dots)$ . For  $n \leq \omega$ ,  $p^{(n)}$  is a type of an  $A$ -indiscernible sequence of length  $n$ .

**Fact 4.10.** [HP11, Lemma 2.5] *If  $T$  is NIP then for a set  $A$  the map  $p(x_0) \mapsto p^{(\omega)}(x_0, \dots)|_A$  from global  $A$ -invariant types to  $\omega$ -types over  $A$  is injective.*

In the rest of the section,  $\Delta$  will always denote a finite set of formulas, closed under negation.

*Claim 4.11.* For every set  $A \subseteq C$ , any type  $q(x) \in S_\Delta(\mathfrak{C})$  which is finitely satisfiable in  $A$  and any choice of a co-heir  $q' \in S(\mathfrak{C})$  over  $A$  which completes  $q$ :

- $(a_0, \dots, a_{n-1}) \models (q'^{(n)})|_C \upharpoonright \Delta$  if and only if  $a_0 \models q|_C$ ,  $a_1 \models q|_{C \setminus a_0}$ , etc.

This enables us to define  $q^{(n)}(x_0, \dots, x_{n-1}) \in S_\Delta(\mathfrak{C})$  as  $q^{(n)} \upharpoonright \Delta$ .

It follows that  $q^{(n)}$  is a type of a  $\Delta$ -indiscernible sequence of length  $n$ .

*Proof.* The proof is by induction on  $n$ :

Right to left: suppose  $a_i \models q|_{C_{a_0, \dots, a_{i-1}}}$  for  $i \leq n$ ,  $\varphi(x_0, \dots, x_n, y) \in \Delta$  and  $\varphi(x_0, \dots, x_n, c) \in q^{(n+1)}$  for  $c \in C$  but  $\neg\varphi(a_0, \dots, a_n, c)$  holds. Then by the choice of  $a_n$ ,  $\neg\varphi(a_0, \dots, a_{n-1}, x, c) \in q$ . Suppose  $(b_0, \dots, b_{n-1}) \models q^{(n)}|_C$ , then  $\varphi(b_0, \dots, b_{n-1}, x, c) \in q$  so there is some  $c' \in A$  such that  $\varphi(b_0, \dots, b_{n-1}, c', c) \wedge \neg\varphi(a_0, \dots, a_{n-1}, c', c)$  holds. But this is a contradiction to the induction hypothesis.

Left to right is similar. □

The following is a local version of Fact 4.10, which will be useful later:

**Proposition 4.12.** (*T dependent*) Suppose  $\Delta$  is a finite set of formulas,  $x$  a finite tuple of variables. Then there exists  $n < \omega$  and finite set of formulas  $\Delta_0$  such that for every set  $A$ , if  $q_1(x), q_2(x) \in S(\mathfrak{C})$  are co-heirs over  $A$  and  $(q_1^{(n)} \upharpoonright \Delta_0)|_A = (q_2^{(n)} \upharpoonright \Delta_0)|_A$  then  $q_1 \upharpoonright \Delta = q_2 \upharpoonright \Delta$ .

*Proof.* By compactness and NIP,

- there exists some finite set of formulas  $\Delta_0$  and some  $n$  such that for all  $\varphi(x, y) \in \Delta$  and all  $\Delta_0$ -indiscernible sequences  $\langle a_0, \dots, a_{n-1} \rangle$ , there is no  $c$  such that  $\varphi(a_i, c)$  holds if and only if  $i$  is even. We may assume that  $\Delta \subseteq \Delta_0$ .

By Claim 4.11, we can conclude:

- Suppose that  $(q_1^{(n)} \upharpoonright \Delta_0)|_A = (q_2^{(n)} \upharpoonright \Delta_0)|_A$ , but  $q_1 \upharpoonright \Delta \neq q_2 \upharpoonright \Delta$ . Then there is some formula  $\varphi(x, y) \in \Delta$  and some  $c \in \mathfrak{C}$  such that  $\varphi(x, c) \in q_1$  and  $\neg\varphi(x, c) \in q_2$ .  
Since  $(q_1 \upharpoonright \Delta_0)^{(n)}|_A = (q_2 \upharpoonright \Delta_0)^{(n)}|_A$ ,  $(q_1 \upharpoonright \Delta_0)^{(m)}|_A = (q_2 \upharpoonright \Delta_0)^{(m)}|_A$  for every  $m \leq n$ , and it follows by induction on  $m$  that the sequence defined by  $a_0 \models (q_1 \upharpoonright \Delta_0)|_{Ac}$ ,  $a_1 \models (q_2 \upharpoonright \Delta_0)|_{Ac a_0}$ ,  $a_2 \models (q_1 \upharpoonright \Delta_0)|_{Ac a_0 a_1}$ ,  $\dots$ ,  $a_{m-1} \models (q_i \upharpoonright \Delta_0)|_{Ac a_0 \dots a_{m-2}}$  ( $i \in \{1, 2\}$ ) realizes this type. But this entails a contradiction, because  $\langle a_0, \dots, a_{n-1} \rangle$  is a  $\Delta_0$  indiscernible sequence (even over  $A$ ), while  $\varphi(a_i, c)$  holds if and only if  $i$  is even. □

**Problem 4.13.** Does Proposition 4.12 hold for invariant types (not just for co-heirs)?

4.1.3. *Large directionality and definability.* Let us recall the definition of  $\text{ded } \lambda$ .

**Definition 4.14.** Let  $\text{ded } \lambda$  be the supremum of the set

$$\{ |I| \mid I \text{ is a linear order with a dense subset of size } \leq \lambda \}.$$

**Fact 4.15.** It is well known that  $\lambda < \text{ded } \lambda \leq (\text{ded } \lambda)^{\aleph_0} \leq 2^\lambda$ . If  $\lambda^{<\lambda} = \lambda$  then  $\text{ded } \lambda = 2^\lambda$  so  $\text{ded } \lambda = (\text{ded } \lambda)^{\aleph_0} = 2^\lambda$ .

For more, see e.g. [CKS12, Section 6].

**Definition 4.16.** Suppose  $M$  is a model and  $p \in S(M)$ . Let  $M_p$  be  $M$  enriched with externally definable sets defined over a realization. Namely, we enrich the language to a language  $L_p$  by adding new relation symbols  $\{d_p x \varphi(x, y) \mid \varphi(x, y) \text{ is a formula}\}$  (so  $d_p$  is thought of as a quantifier over  $x$ ), and let  $M_p$  be a model for  $L_p$  with universe  $M$  and let  $d_p x \varphi(x, y)$  be interpreted as  $\{b \in M \mid \varphi(x, b) \in p\}$ .

*Remark 4.17.* Every model  $N \models \text{Th}(M_p)$  gives rise to an  $L$  complete type over  $N$ , namely  $p^N = \{\varphi(b, x) \mid b \in N, N \models d_p x \varphi(x, b)\}$ .

*Claim 4.18.* Suppose  $p \in S(M)$ ,  $q \in \text{uf}(p)$ , and  $\bar{a} = \langle a_0, a_1, \dots \rangle \models q^{(\omega)}|_M$ . If  $\text{tp}(\bar{a}/M)$  is not definable with parameters in  $M_p$ , then  $T$  is large.

Moreover, in this case

⊗ There exists a finite  $\Delta$  such that for every  $\lambda \geq |T|$ ,

$$\text{ded } \lambda \leq \sup \{|\text{uf}_\Delta(p)| \mid p \in S(N), |N| = \lambda\}.$$

*Proof.* We may assume that  $|M| = |L|$ : let  $r = q^{(\omega)}|_M$ , and  $N \prec M_r$ ,  $|N| = |L|$ . Then  $N$  gives rise to a complete type  $r'(x_0, x_1, \dots) \in S(N)$ . Let  $p' = r' \upharpoonright x_0$ . It is easy to see that  $r' = q'^{(\omega)}|_N$  for some  $q' \in \text{uf}(p')$ . Also,  $r'$  is not definable with parameters in  $N_{p'}$ .

Let us recall a theorem from [She71a] (we formulate it a bit differently):

Suppose  $L$  is a language of cardinality at most  $\lambda$ ,  $P$  a new predicate (or relation symbol), and  $S$  a complete theory in  $L(P)$ .

**Definition.**  $\text{df}_{\text{iso}} \lambda$  is the supremum of the set of cardinalities

$$|\{B' \subseteq M \mid (M, B') \cong (M, B)\}|$$

where  $(M, B)$  is an  $L(P)$  model of  $S$  of cardinality  $\lambda$ .

**Theorem.** [She71a, Hod93, Theorem 12.4.1] *The following are equivalent<sup>2</sup>:*

- (1)  $P$  is not definable with parameters in  $S$ , i.e. there is no  $L$ -formula  $\theta(x, y)$  such that  $S \models \exists y \forall x (P(x) \leftrightarrow \theta(x, y))$ .
- (2) For every  $\lambda \geq |L|$ ,  $\text{df}_{\text{iso}}(\lambda) \geq \text{ded } \lambda$ .

Let  $n < \omega$  be the first such that  $\text{tp}(\bar{a} \upharpoonright n/M)$  is not definable with parameters in  $M_p$ . So  $1 < n$  and  $r = \text{tp}(a_0, \dots, a_{n-2}/M)$  is definable but  $\text{tp}(a_0, \dots, a_{n-1}/M)$  is not.

---

<sup>2</sup>The original theorem referred to  $\text{ded}^* \lambda$ , which counts the number of branches of the same height in a tree with  $\lambda$  many nodes, but it equals  $\text{ded } \lambda$ , see [CKS12, Section 6].

For a formula  $\alpha(x_0, \dots, x_{n-2}, y)$  let  $(d_r \alpha)(y)$  be a formula in  $L(M_p)$  defining  $\alpha(\bar{a} \upharpoonright n-1, M)$ . If  $M_p \prec N \models T_p$  then, as in Remark 4.17, there is a complete  $L$  type  $r^N(x_0, \dots, x_{n-2})$  over  $N$  defined by  $\alpha(x_0, \dots, x_{n-2}, b) \in r^N$  if and only if  $N \models (d_r \alpha)(b)$ .

There is some formula  $\varphi(x_0, \dots, x_{n-1}, y)$  such that the set  $B_0 := \varphi(a_0, \dots, a_{n-1}, M)$  is not definable with parameters in  $M_p$ . Let  $S = \text{Th}(M_p, B_0)$  in the language  $L(M_p)(P)$  (naming elements from  $M$ , so that  $N \models S$  implies  $M_p \prec N$ ).

By the theorem cited above, for every  $\lambda \geq |L|$  and  $\kappa < \text{ded } \lambda$ , there exists a model  $(N_{\lambda, \kappa}, B_{\lambda, \kappa}) = (N, B) \models S$  of cardinality  $\lambda$  such that, letting  $\mathcal{B}^N = \{B' \mid (N, B') \cong (N, B)\}$ ,  $|\mathcal{B}^N| \geq \kappa$ .

Let  $\bar{a}^N = (a_0^N, \dots, a_{n-2}^N) \models r^N$  and for every  $B' \in \mathcal{B}^N$ , let

$$q_{B'} = p^N(x) \cup \{\varphi(\bar{a}^N, x, \bar{b}) \mid \bar{b} \in B'\} \cup \{\neg \varphi(\bar{a}^N, x, \bar{b}) \mid \bar{b} \notin B'\}.$$

By choice of  $S$ ,  $B$  and  $\mathcal{B}$ ,  $q_{B'}$  is finitely satisfiable in  $N$  and for  $B' \neq B'' \in \mathcal{B}^N$ ,  $q_{B'} \upharpoonright \varphi \neq q_{B''} \upharpoonright \varphi$ , so now  $N \upharpoonright L$  is a model of  $T$  with a type  $p^N$  such that  $|\text{uf}_{\{\varphi\}}(p)| \geq \kappa$ .  $\square$

If  $T$  is small we can say more:

*Claim 4.19.* Assume  $T$  is small. Suppose  $p \in S(M)$ ,  $q \in \text{uf}(p)$ , and  $\bar{a} = \langle a_0, a_1, \dots \rangle \models q^{(\omega)}|_M$ , then  $\text{tp}(\bar{a}/M)$  is definable over  $\text{acl}^{\text{eq}}(\emptyset)$  in  $M_p$ .

*Proof.* By Claim 4.18 it is definable with parameters in  $M_p$ . Let  $n < \omega$  be minimal such that  $q^{(n)}|_M$  is not definable over  $\text{acl}^{\text{eq}}(\emptyset)$  in  $M_p$ . Suppose that for some formula  $\varphi(x_0, \dots, x_{n-1}, y)$ ,  $\text{tp}_\varphi(\bar{a}/M)$  is not definable over  $\text{acl}^{\text{eq}}(\emptyset)$  in  $M_p$ . This means that while the set  $\{b \in M \mid \mathfrak{C} \models \varphi(\bar{a}, b)\}$  is definable by  $\psi(y, c)$  for some  $\psi$  in  $L(M_p)$ ,  $c \notin \text{acl}^{\text{eq}}(\emptyset)$ . We may assume that  $c \in M_p^{\text{eq}}$  is the code of this set (for every automorphism  $\sigma$ ,  $\sigma$  fixes this set if and only if  $\sigma(c) = c$ ). So in some elementary extension  $M_p \prec N$ , there are infinitely many conjugates of  $c$ ,  $\{c_i \mid i < \omega\}$ , such that  $\psi(N, c_i) \neq \psi(N, c_j)$  for  $i \neq j$ . This implies that  $\text{uf}_{\{\varphi\}}(p^N) \geq \aleph_0$ , just as in the proof of Claim 4.18.  $\square$

**Corollary 4.20.** ( $T$  dependent)  $T$  is large if and only if for every  $\lambda \geq |T|$ ,

$$\sup\{|\text{uf}_\Delta(p)| \mid p \in S(M), \Delta \text{ finite}, |M| = \lambda\} = \text{ded}(\lambda).$$

*Proof.* Suppose  $T$  is large, i.e. for some  $\Delta$ ,  $M$ ,  $p \in S(M)$  and  $\lambda \geq M$ ,  $|\text{uf}_\Delta(p)| > \lambda$ . By Proposition 4.12, find some  $\Delta_0$  and  $n$  such that

$$|\text{uf}_\Delta(p)| \leq \left| \left\{ \left( q^{(n)} \upharpoonright \Delta_0 \right) |_M \mid q \in \text{uf}(p) \right\} \right|.$$

Hence there is some  $q \in \text{uf}_\Delta(p)$  such that  $q^{(n)} \upharpoonright \Delta_0$  is not definable with parameters over  $M_p$ , and we are done by Claim 4.18 (also note that  $|S_\Delta(M)| \leq \text{ded } |M|$  for every finite  $\Delta$  in dependent theories (see e.g. [She71b, Theorem 4.3])).  $\square$

#### 4.1.4. Trichotomy theorem.

**Theorem 4.21.** *For every  $T$ ,  $T$  is either small, medium or large. Moreover,*

- (1)  $T$  is small iff for all  $M \models T$ ,  $p(x) \in S(M)$ ,  $|uf(p)| \leq 2^{|T|+|lg(x)|}$ .
- (2)  $T$  is medium iff for all  $M \models T$ ,  $p(x) \in S(M)$ ,  $|uf(p)| \leq |M|^{|T|+|lg(x)|}$  and  $T$  is not small.
- (3)  $T$  is large iff  $T$  is not small nor medium.

*Proof.* (1) is Claim 4.6. (3) is the definition.

(2) Left to right is clear. Conversely, since  $T$  is not small, by Claim 4.6,

$$\sup\{|uf_{\Delta}(p)| \mid p \in S(M), \Delta \text{ finite}, |M| = \lambda\} \geq \lambda.$$

On the other hand, if it is strictly greater than  $\lambda$ , then by the proof of Corollary 4.20,  $T$  is large.  $\square$

In Section 4.2, we will show that these classes are not empty, and thus:

**Corollary 4.22.** *For  $\lambda \geq |T|$ , the cardinality*

$$\sup\{|uf_{\Delta}(p)| \mid p \in S(M), \Delta \text{ finite}, |M| = \lambda\}$$

*has four possibilities: finite /  $\aleph_0$  ;  $\lambda$ ;  $\text{ded } \lambda$ ;  $2^{2^\lambda}$ .*

*This corresponds to small, medium, and large directionality (the last one happens when the theory has the independence property, see Observation 4.4).*

**Problem 4.23.** Suppose  $T$  is interpretable in  $T'$ , and  $T$  is large. Does this imply that  $T'$  is large?

**4.2. Examples of different directionalities.** Here we give examples of the different directionalities.

##### 4.2.1. Small directionality.

**Example 4.24.**  $\text{Th}(\mathbb{Q}, <)$  has small directionality. In fact, every 1-type over a model  $M$ , has at most 2 global co-heirs, and in general, a type  $p(x_0, \dots, x_{n-1})$  is determined by the order type of  $\{x_0, \dots, x_{n-1}\}$  and  $p \upharpoonright x_0, p \upharpoonright x_1$ , etc.

**Proposition 4.25.** *The theory of dense trees is also small.*

*Proof.* So here  $T$  is the model completion of the theory of trees in the language  $\{<, \wedge\}$ .

*Claim.* Let  $M \models T$  and  $p(x_0, x_1, \dots, x_{n-1}) \in S(M)$  be any type. Then  $\bigcup_{i,j < n} p \upharpoonright (x_i, x_j) \cup p|_{\emptyset} \vdash p$ .

*Proof.* Let  $\Sigma = \bigcup_{i,j < n} p \upharpoonright (x_i, x_j) \cup p|_{\emptyset}$ . Suppose  $(a_0, \dots, a_{n-1}) \models \Sigma$ . By quantifier elimination, the formulas in  $p$  are Boolean combination of formulas of the form  $\bigwedge_{k < m} x_{j_k} \wedge a \leq \bigwedge_{l < r} x_{r_l} \wedge b$  where  $b, a \in M$  and  $j_0, \dots, j_{k-1}, r_0, \dots, r_{l-1} < n$ .

If  $\mathbf{a}, \mathbf{b}$  does not appear,  $\bar{\mathbf{a}}$  satisfy this formula because we included  $\mathbf{p}|_{\emptyset}$ . Consider  $\bigwedge_{k < m} x_{j_k} \wedge \mathbf{a}$ : by assumption we know what is the ordering of  $\{x_{j_k} \wedge \mathbf{a} \mid k < m\}$  (this set is linearly ordered — it is below  $\mathbf{a}$ ). Hence, as  $\bar{\mathbf{a}} \models \Sigma$ ,  $\bigwedge_{k < m} x_{j_k} \wedge \mathbf{a}$  must be equal to the minimal element in this set, namely  $x_{j_k} \wedge \mathbf{a}$  for some  $k$ , which is determined by  $\Sigma$ . Now  $x_{j_k} \wedge \mathbf{a} \leq \bigwedge_{l < r} x_{r_l} \wedge \mathbf{b}$  holds if and only if for each  $r < l$ , we have  $x_{j_k} \wedge \mathbf{a} \leq x_{r_l}$  and  $x_{j_k} \wedge \mathbf{a} \leq \mathbf{b}$ , both decided in  $\Sigma$ .

Note that we can get rid of  $\mathbf{p}|_{\emptyset}$  but we should replace 2-types by 3-types.  $\square$

*Claim.* For any  $A \subseteq M \models T$ , and  $\mathbf{p}(x), \mathbf{q}(y) \in S(A)$ , there are only finitely many complete type  $\mathbf{r}(x, y)$  that contain both  $\mathbf{p}$  and  $\mathbf{q}$ . In fact there is a uniform bound on their number.

*Proof.* We may assume that  $A$  is a substructure. For any  $\mathbf{a} \in M$ , the structure generated by  $A$  and  $\mathbf{a}$ , denoted by  $A(\mathbf{a})$ , is just  $A \cup \{\mathbf{a}_A, \mathbf{a}\}$  where  $\mathbf{a}_A = \max\{\mathbf{b} \wedge \mathbf{a} \mid \mathbf{b} \in A\}$ . Note that  $\mathbf{a}_A$  need not exist, but if it does, then it is the only new element except  $\mathbf{a}$  (because if  $\mathbf{b}_1 \wedge \mathbf{a} < \mathbf{b}_2 \wedge \mathbf{a}$  then  $\mathbf{b}_1 \wedge \mathbf{a} = \mathbf{b}_1 \wedge \mathbf{b}_2$ ).

Now, let  $\mathbf{a} \models \mathbf{p}$  and  $\mathbf{b} \models \mathbf{q}$ . Let  $B = B(\mathbf{p}, \mathbf{q}) = \{\mathbf{d} \in A \mid \mathbf{d} \leq \mathbf{a} \text{ \& \& } \mathbf{d} \leq \mathbf{b}\}$ . This set is linearly ordered, and it may have a maximum. If it does, denote it by  $\mathbf{m}$ . Note that  $\mathbf{m}$  depends only on  $\mathbf{p}$  and  $\mathbf{q}$ .

Now it is easy to show that  $\text{tp}(\mathbf{a}, \mathbf{a}_A, \mathbf{b}, \mathbf{b}_A, \mathbf{a} \wedge \mathbf{b} / \mathbf{m}) \cup \mathbf{p}(x) \cup \mathbf{q}(y)$  determines  $\text{tp}(\mathbf{a}, \mathbf{b} / A)$  by quantifier elimination. This suffices because the number of types of finite tuples over a finite set is finite.  $\square$

Let  $M \models T$ ,  $\mathbf{p} \in S(M)$  and  $I$  be the set of all types  $\mathbf{r}(\bar{x}_0, \bar{x}_1, \dots)$  over  $M$  such that realizations of  $\mathbf{r}$  are indiscernible sequences of tuples satisfying  $\mathbf{p}$ .

Let  $V = \{(y_0, y_1) \mid y_0, y_1 \text{ are any 2 variables from } \bar{x}_0, \bar{x}_1\}$ . By the second claim, for any  $(y_0, y_1) \in V$ , the set  $\{\mathbf{r} \upharpoonright (y_0, y_1) \mid \mathbf{r} \in I\}$  is finite. By the first claim and indiscernibility, the function taking  $\mathbf{r}$  to  $(\mathbf{r}|_{\emptyset}, \langle \mathbf{r} \upharpoonright (y_0, y_1) \mid (y_0, y_1) \in V \rangle)$  is injective. Together, it means that  $|I| \leq 2^{\aleph_0}$  and we are done by Fact 4.10.  $\square$

#### 4.2.2. Medium directionality.

**Example 4.26.** Let  $L = \{P, Q, H, <\}$  where  $P$  and  $Q$  are unary predicates,  $H$  is a unary function symbol and  $<$  is a binary relation symbol. Let  $T^\forall$  be the following theory:

- $P \cap Q = \emptyset$ .
- $H$  is a function from  $P$  to  $Q$  (So  $H \upharpoonright Q = \text{id}$ ).
- $(P, <, \wedge)$  is a tree.

And let  $T$  be its model completion (so  $T$  eliminates quantifiers). Note that there is no structure on  $Q$ . So as in Section 2,  $T$  is dependent (this theory is interpretable in the theory there).

Let  $T'$  be the restriction of  $T$  to the language  $L' = L \setminus \{H\}$ . The same “moreover” part applies here as in Corollary 2.6, so  $T'$  is the model completion of  $T^\forall \upharpoonright L'$  and also eliminates quantifiers.



*Claim 4.27.*  $T'$  has small directionality.

*Proof.* The only difference between  $T'$  and dense trees is the new set  $Q$  which has no structure. Easily this does not make any difference.  $\square$

**Proposition 4.28.**  $T$  has medium directionality.

*Proof.* Let  $M \models T$ . Let  $B$  be a branch in  $P^M$  (i.e. a maximal linearly ordered set). Let  $p(x) \in S(M)$  be a complete type containing  $\Sigma_B := \{b < x \mid b \in B\}$ . Note that  $\Sigma$  “almost” isolates  $p$ : the only freedom we have, is to determine what is  $H(x)$ . So suppose  $H(x) = m \in p$  for  $m \in Q$ .

Let  $c \models p$  (so  $c \notin M$ ). For each  $a \in Q^M$ , let  $p_a(x) = p \cup \{H(c \wedge x) = a\}$ .

Then  $p_a$  is finitely satisfiable in  $M$ : Suppose  $\Gamma \subseteq p_a$  is finite. By quantifier elimination, we may assume that  $\Gamma \subseteq \Sigma_B \cup \{H(c \wedge x) = a\} \cup \{H(x) = m\}$ . Since  $B$  is linearly ordered, we may assume that  $\Gamma = \{b < x, H(c \wedge x) = a, H(x) = m\}$  for some  $b \in B$ . Since  $T$  is the model completion of  $T^\forall$  which has the amalgamation property, there are two elements  $d, e \in M$  such that  $b < d, e, e \in B, d \notin B, H(d) = m$  and  $H(d \wedge e) = a$ . Since  $d \wedge c = d \wedge e, d \models \Gamma$ . We have found  $|Q^M|$  co-heirs of  $p$ , and since  $M$  was arbitrary  $T$  is not small.

This gives a lower bound on the directionality of  $T$ , and we would like to find an upper bound as well. We shall use the same idea as in the proof of Proposition 4.25.

Let  $M \models T, p \in S(M)$  and  $I$  be the set of all types  $r(\bar{x}_0, \bar{x}_1, \dots)$  over  $M$  such that realizations of  $r$  are indiscernible sequences of tuples satisfying  $p$ . Let  $I' = \{r \restriction L' \mid r \in I\}$ . By the proof of proposition 4.25,  $|I'| \leq 2^{\aleph_0}$ .

Let  $V = \{t \mid t \text{ is a term in } L \text{ in the variables } \bar{x}_0, \bar{x}_1, \dots \text{ over } \emptyset\}$ . Suppose  $r \in I$ , then, as in the proof of Proposition 4.25, for every term  $t$  such that  $P(t) \in r$ , let  $t_r = \max\{a \wedge t \mid a \in M\}$  — a term over  $M$  (it need not exist). To determine  $r$ , it is enough to determine the equations that occur between the images under  $H$  of the  $t_r$ ’s and the  $t$ ’s over  $M$ . This shows that  $|I| \leq |M|^{\aleph_0}$ .  $\square$

#### 4.2.3. Large directionality.

**Example 4.29.** Let  $L = \{P, Q, H, <_P, <_Q\}$  where  $P$  and  $Q$  are unary predicates,  $H$  is a unary function symbol and  $<_P, <_Q$  are binary relation symbols. Let  $T^\forall$  be the following theory:

- $P \cap Q = \emptyset$ .
- $H$  is a function from  $P$  to  $Q$ .
- $(P, <_P, \wedge)$  is a tree.
- $(Q, <_Q)$  is a linear order.

**Proposition 4.30.**  $T$  has large directionality.

*Proof.* This is similar to the proof of Proposition 4.28.

Let  $M \models T$ . Let  $B$  be a branch in  $P^M$ . Let  $p(x) \in S(M)$  be a complete type containing  $\Sigma_B := \{b < x \mid b \in B\}$  saying that  $H(x) = m$  for some  $m \in Q^M$ .

Let  $c \models p$  (so  $c \notin M$ ). For each cut  $I \subseteq Q^M$ , let

$$p_I(x) = p \cup \{e < H(c \wedge x) < f \mid e \in I, f \in Q^M \setminus I\}.$$

Then  $p_I$  is finitely satisfiable in  $M$  as in the proof of 4.28. So for every cut in  $Q$  we found a co-heir of  $p$ , and since  $M$  was arbitrary  $T$  is not small nor medium (because for every linear order, we can find a model such that  $Q$  contains this order).  $\square$

The above example can be transformed to an example in valued fields:

**Definition 4.31.** The language  $L$  of valued fields is the following. It is a 3-sorted language, one sort for the base field  $K$  equipped with the ring language  $\{0, 1, +, \cdot\}$ , another for the valuation group  $\Gamma$  equipped with the ordered abelian groups language  $L_\Gamma = \{0, +, <\}$ , and another for the residue field  $k$  equipped with the ring language  $L_k$ , we also have the valuation map  $v : K^\times \rightarrow \Gamma$ , and an angular component map  $ac : K \rightarrow k$ . Recall that an angular component is a function that satisfies  $ac(0) = 0$  and  $ac \upharpoonright K^\times : K^\times \rightarrow k^\times$  is a homomorphism such that if  $v(x) = 0$  then  $ac(x)$  is the residue of  $x$ .

For more on valued fields with angular component, see e.g. [Bél99, Pas89], which also gives us the following fact

**Fact 4.32.** [Pas89, Theorem 4.1]

- (1) *The theory of Henselian valued fields of characteristic  $(0, 0)$  in the language  $L$  has elimination of field quantifiers, i.e. every formula  $\varphi(x)$  is equivalent to a Boolean combination of formulas of the form  $\varphi(ac(f_0(x)), \dots, ac(f_{n-1}(x)))$  and  $\chi(v(g_0(x)), \dots, v(g_{m-1}(x)))$  where  $\varphi$  is a formula in  $L_k$ ,  $\chi$  is a formula in  $L_\Gamma$ ,  $g_i$  and  $f_j$  are polynomials over the integers.*
- (2) *Moreover, the theory admits cell decomposition which implies that if  $M$  is a model, then every type  $p(x) \in S(M)$  (where  $x \in K$ ) is isolated by the set of formulas*

$$p \cap \left[ \bigcup_{a \in K^M} \{\varphi(ac(x-a)) \mid \varphi \in L_k(k^M)\} \cup \bigcup_{a \in K^M} \{\chi(v(x-a)) \mid \chi \in L_\Gamma(\Gamma^M)\} \right].$$

**Theorem 4.33.** *Let  $T = \text{Th}(K, \Gamma, k)$  be a theory of Henselian valued fields of characteristic  $(0, 0)$ . Then  $T$  has medium or large directionality.*

*Proof.* Let  $M_0 = (K_0, \Gamma_0, k_0) \models T$  be a countable model, and let  $\langle \gamma_i \mid i < \omega \rangle$  be an increasing sequence of elements from  $\Gamma^+$ . Let  $S = {}^{\omega \geq 2}$ ,  $S_0 = {}^{\omega > 2}$  and we give  $S$  (and  $S_0$ ) a tree structure  $(<_S, \wedge_S)$ . Let  $f : \{(x, y) \in S_0^2 \mid x < y\} \rightarrow k_0$  be some function onto  $k_0$  such that:

- $s_1 < s_2 < s_3 \Rightarrow f(s_1, s_2) = f(s_1, s_3)$ ,
- for each  $x \in k_0$ , and each branch  $\eta \in {}^\omega 2$ ,  $\{s \in S_0, s < \eta \mid \exists t > s, t < \eta [f(s, t) = x]\}$  is infinite, and
- if  $t_1 \wedge t_2 = s$  then  $f(s, t_1) \neq f(s, t_2)$ .

Let  $\Sigma(x_s \mid s \in S)$  be the following set of formulas with variables in the field sort (over  $M_0$ ):

$$\{v(x_s) = 0 \mid s \in S\} \cup \{v(x_s - x_t) = \gamma_{\text{lev}(s \wedge t)} \mid s, t \in S\} \cup \{\text{ac}(x_t - x_s) = f(s, t) \mid s < t \in S\}.$$

Then  $\Sigma$  is consistent with  $M_0$ , and let  $(M, a_s \mid s \in S)$  be a countable model realizing  $\Sigma$  containing  $M_0$ . For each branch  $\eta \in {}^\omega \omega$ , let  $p_\eta(x)$  be the following type in the valued field sort

$$\{v(x - a_s) \geq \gamma_{\text{lev}(s)} \mid s \in \eta\}.$$

If  $\eta_1 \neq \eta_2$  then  $p_{\eta_1} \neq p_{\eta_2}$  because if  $s = \eta_1 \wedge \eta_2$  and  $s < t < \eta_1$  then  $p_{\eta_1} \models v(x - a_t) = \gamma_{\text{lev}(t)}$  (as  $v(x - a_t) = v(x - a_{t'} + a_{t'} - a_t) = v(a_{t'} - a_t) = \gamma_{\text{lev}(t)}$  for any  $t < t' < \eta_1$ ) while  $p_{\eta_2} \models v(x - a_t) = \gamma_{\text{lev}(s)}$  (because  $p_{\eta_2} \models v(x - a_t) = v(x - a_s + a_s - a_t) \geq \gamma_{\text{lev}(s)}$ , but  $p_{\eta_2} \models \text{ac}(x - a_s) = \text{ac}(a_{t'} - a_s)$  for any  $s < t' < \eta_2$ , and since  $\text{ac}(a_{t'} - a_s) \neq \text{ac}(a_t - a_s)$ , we are done).

Since  $M$  is countable, there is some branch  $\eta \in {}^\omega 2$  such that  $p_\eta$  is not satisfied in  $M$ .

Note that if  $a \in K^M$  then there is some  $s < \eta$  such that  $v(a - a_s) < \gamma_{\text{lev}(s)}$ , but then  $p_\eta \models v(x - a) = v(a_s - a)$ . Also,  $p_\eta \models \text{ac}(x - a) = \text{ac}(a_s - a)$ . By fact 4.32, this means that  $p_\eta$  is a complete type over  $M$ .

Let  $c \models p_\eta$ . Then, for each  $a \in k_0$ ,  $p_a := p(x) \cup \{\text{ac}(c - x) = a\}$  is finitely satisfiable in  $K^M$  (each finite subset is realized by  $a_s$  for some  $s$ ), and if  $a \neq b$  then  $p_a \neq p_b$  and we are done.  $\square$

*Remark 4.34.* In the theorem, instead of assuming that the field is Henselian of characteristic  $(0, 0)$ , it is enough to assume that it eliminates field quantifiers and that the residue field is infinite. Note that by [Yin09], if the valuation group is a  $\mathbb{Z}$ -group, then the field is Henselian.

Indeed, let  $\bar{M} = (\bar{K}, \bar{k}, \bar{\Gamma})$  be the algebraic closure of  $M$  as a valued field. We embed it in a large algebraic closure  $\Omega$  of the monster model  $\mathfrak{C}$  of  $M$ . Since it is still countable, there is some  $\eta \in 2^\omega$  such that  $p_\eta$  is not satisfied even in  $\bar{M}$ . Then for every polynomial  $f(x)$  over  $K$  and every  $s < \eta$  large enough,  $p_\eta \models \text{ac}(f(x)) = \text{ac}(f(a_s)) \wedge v(f(x)) = v(f(a_s))$  (decompose  $f$  into linear factors  $\prod (x - a_i)$ ). For every large enough  $s$ ,  $v(a_i - a_s) < \gamma_{\text{lev}(s)}$  for all  $i$ , and so if  $d \models p_\eta$  in  $\mathfrak{C}$ , then  $\text{ac}(f(d)) = \text{ac}(f(a_s))$  (because  $\text{res}\left(\frac{f(d)}{f(a_s)}\right) = 1$  — we do not assume that  $\text{ac}$  extends to  $\bar{M}$ ) and  $v(f(d)) = v(f(a_s))$ ). So  $p_\eta$  is a complete type over  $M$ , and the proof goes through.

**Problem 4.35.** We do not know what is the directionality in the case of fields where the residue field is finite, such as the  $p$ -adics.

It turns out that even RCF has large directionality, as we shall present now.

Apparently, that RCF was not small was already known and can be deduced from Marcus Tressl's thesis (see [Tre96, 18.13]), but here we give a direct proof that RCF is in fact large and even more.

**Definition 4.36.** Let  $M \models \text{RCF}$ . A type  $p \in S(M)$  is called dense if it is not definable and the differences  $b - a$  with  $a, b \in M$  and  $a < x < b \in p$ , are arbitrarily (w.r.t.  $M$ ) close to 0.

For example, if  $R$  is the real closure of  $\mathbb{Q}$ , then  $\text{tp}(\pi/R)$  is dense.

**Fact 4.37.** *Any real closed field can be embedded into a real closed field of the same cardinality with some dense type.*

*Proof.* [due to Marcus Tressl] Let  $R$  be a real closed field. Let  $S$  be the (real closed) field  $R((t^{\mathbb{Q}}))$  of generalized power series over  $R$  with coefficients in the exponents  $\mathbb{Q}$ . Let  $K$  be the definable closure of  $R(t)$  in  $S$  and let  $p$  be the 1-type of the formal Taylor series of  $e^t$  over  $K$ :  $\text{tp}(1 + t^1/1! + t^2/2! + \dots/K)$ . Then  $p$  is a dense 1-type over  $K$ .  $\square$

*Claim 4.38.* Suppose  $p$  is dense and  $q$  is a definable type over  $M \models \text{RCF}$  and both are complete. Then  $q$  and  $p$  are weakly orthogonal, meaning that  $p(x) \cup q(y)$  implies a complete type over  $M$ .

*Proof.* [Remark: this is an easy result that is well known, but for completeness we give a proof.]

Let  $\omega \models q, \alpha \models p$ .

Note that since  $p$  is not definable over  $M$ , for every  $m \in M$ , and even for every  $m$  such that  $\text{tp}(m/M)$  is definable, there is some  $\varepsilon_m \in M$  such that  $0 < \varepsilon_m < |\alpha - m|$ .

Now, suppose that  $\varphi(x, y)$  is any formula over  $M$ . Then, as  $q$  is definable, there is a formula  $\psi(x) := (d_q y) \varphi(x, y)$  over  $M$  that defines  $\varphi(M, \omega)$ . We claim that  $\varphi(\alpha, \omega)$  if and only if  $\psi(\alpha)$  (i.e. if  $p(x) \cup q(y)$  then  $\varphi(x, y)$  if and only if  $(d_q y) \varphi(x, y) \in p$ ).

We know that  $\psi$  is equivalent to a finite union of intervals and points from  $M$ . We also know that  $\varphi(\mathfrak{C}, \omega)$  is such a union, but the types of the end-points over  $M$  are definable over  $M$  (since we have definable Skolem functions). So denote the set of all these end-points by  $A$ . Let  $0 < \varepsilon \in M$  be smaller than every  $\varepsilon_m$  for each  $m \in A$ . Let  $a, b \in M$  such that  $a < \alpha < b$  and  $b - a < \varepsilon$ . Then necessarily

- $\psi(\alpha)$  holds if and only if
- $\psi(m)$  holds for all / some  $m \in M$  such that  $a \leq m \leq b$  if and only if
- $\varphi(m, \omega)$  holds for all / some  $m \in M$  such that  $a \leq m \leq b$  if and only if
- $\varphi(\alpha, \omega)$  holds.

$\square$

We claim that RCF has large directionality. Moreover, we seem to answer an open question raised in [Del84], as she says there:

Mais il laisse ouverte la possibilité que la borne du nombre de cohéritiers soit  $\text{ded } |M|$  dans le cas de la propriété de l'ordre et  $(\text{ded } |M|)^{(\omega)}$  dans le cas de l'ordre multiple.

So let us make clear what the question means:

**Definition 4.39.** (Taken from [Kei78, Kei76]) A theory  $T$  is said to have the *multiple order property* if there are formulas  $\varphi_n(x, y_n)$  for  $n < \omega$  such that the following set of formulas is consistent with  $T$ :

$$\Gamma = \left\{ \varphi_n(x_\eta, y_{n,k})^{\eta(k) < n} \mid \eta \in {}^\omega \omega \right\}.$$

*Remark 4.40.* If  $T$  is strongly dependent (see 2.3), for example, if  $T = \text{RCF}$ , it does not have the multiple order property.

*Proof.* Suppose  $T$  has the multiple order property as witnessed by formulas  $\varphi_n$ . Consider the formulas  $\psi_n(x, y, z) = \varphi_n(x, y) \leftrightarrow \varphi_n(x, z)$ . It is easy to see that  $\{\psi_n \mid n < \omega\}$  ex amplifies that the theory is not strongly dependent.  $\square$

**Fact 4.41.** [Kei78] *If  $T$  is countable and has the multiple order property, then for every  $\kappa$ ,  $\sup\{S(M) \mid M \models T, |M| = \kappa\} \geq (\text{ded } \kappa)^\omega$ . If  $T$  does not have the multiple order property, then  $\sup\{S(M) \mid M \models T, |M| = \kappa\} \leq \text{ded } \kappa$  for every  $\kappa$ .*

So the question can be formulated as follows:

- Is there a countable theory without the multiple order property such that for every  $\kappa \geq \aleph_0$ ,  $\sup\{\text{uf}(p) \mid p \in S(M), |M| = \kappa\} = (\text{ded } \kappa)^\omega$ .

It is a natural question, because of 2 reasons:

- (1) In general, the number of types (in  $\alpha$  variables) over a model of size  $\lambda$  in a dependent theory is bounded by  $(\text{ded } \lambda)^{|\Gamma| + |\alpha| + \aleph_0}$  (by [She71b, Theorem 4.3]), so by Fact 4.10 this is an upper bound for  $\sup\{\text{uf}(p) \mid p \in S(M), |M| = \kappa\}$ .
- (2) It is very easy to construct an example with the multiple order property that attains this maximum: for example, one can modify example 4.29, and add independent orderings to  $Q$ .

**Theorem 4.42.** *RCF has large directionality. Moreover, RCF does not have the multiple order property but for every  $\lambda \geq \aleph_0$ ,*

$$\sup\{\text{uf}(p) \mid p \in S(M), |M| = \lambda\} \geq (\text{ded } \lambda)^\omega.$$

*Proof.* Let  $K \models \text{RCF}$  with a dense type  $p$  over it. Let  $\alpha \models p$ , and let  $\omega$  be an element greater than any element in  $K$ . Then  $q = \text{tp}(\omega/K)$  is definable and we can apply Claim 4.38. Let  $p(x_\omega, x_\alpha) = \text{tp}(\omega, \alpha/K)$ .

For every sequence  $\bar{I} = \langle I_i \mid i < \omega \rangle$  of bounded first segments  $I_i \subseteq K$ , we define a sequence of terms: for every pair of sequences  $\bar{a} = \langle a_i \mid i < \omega \rangle$  and  $\bar{b} = \langle b_i \mid i < \omega \rangle$  such that  $a_i \in I_i$  and  $b_i \notin I_i$ , and for each  $n < \omega$  let  $f_n(\bar{a}, x) = \alpha + \sum_{i=0}^n (a_i/x^i)$  and  $g_n(\bar{a}, \bar{b}, x) = \alpha + \sum_{i=0}^{n-1} (a_i/x^i) + b_n/x^n$ .

Now, let  $p_{\bar{I}}(x_\omega, x_\alpha)$  be

$$p_{\bar{I}}(x_\omega, x_\alpha) = p(x_\omega, x_\alpha) \cup \{f_n(\bar{a}, x_\omega) < x_\alpha < g_n(\bar{a}, \bar{b}, x_\omega) \mid \bar{a}, \bar{b} \text{ as above, } n < \omega\}.$$

Then this type (which is over  $K \cup \{\alpha\}$ ) is finitely satisfiable in  $K$  because of the weak orthogonality. For  $\bar{I}, \bar{J}$ , two different sequences of first segments,  $p_{\bar{I}}$  and  $p_{\bar{J}}$  contradict each other. Since  $K$  was arbitrary, we are done by Fact 4.37:

Suppose first that  $\text{ded } \lambda$  is attained. I.e. there is a linear order  $I$  of size  $\lambda$  with  $\text{ded } \lambda$  many cuts. There is a linear order of size  $\lambda$  having  $\omega$  disjoint copies of  $I$  as substructures. So we can embed this order in a real closed field  $R$  of size  $\lambda$ , which we can embed again in a real closed field  $K$  with a dense type and the same cardinality, and apply the above.

If  $\text{ded } \lambda$  is not attained, then the cofinality of  $\text{ded } \lambda$  is greater than  $\lambda$ , so  $(\text{ded } \lambda)^\omega = \sup\{\kappa^\omega \mid \kappa < \text{ded } (\lambda)\}$ , so a similar argument applies.  $\square$

Marcus Tressl has pointed out the type  $\text{tp}(\alpha, \omega)$  to us as a type with infinitely many co-heirs (this follows from [Tre96, 18.13]). We thank him for that. The explanation, and the proof that the theory is large is ours.

## 5. SPLINTERING

This part of the paper is motivated by the work of Rami Grossberg, Andrés Villaveces and Monica VanDieren. In their paper [GAV]. They were studying Shelah's Generic pair conjecture (which is now a theorem — [She, She12, She11]) and in their analysis, they came up with the notion of splintering, a variant of splitting.

**Definition 5.1.** Let  $p \in S(\mathfrak{C})$ . Say that  $p$  *splinters* over  $M$  if there is some  $\sigma \in \text{Aut}(\mathfrak{C})$  such that

- (1)  $\sigma(p) \neq p$ .
- (2)  $\sigma(p|_M) = p|_M$ .
- (3)  $\sigma(M) = M$  setwise.

*Remark 5.2.* [due to Martin Hils] Splitting implies splintering, and if  $T$  is stable, then they are equal.

*Proof.* Suppose  $p \in S(\mathfrak{C})$  does not split over  $M$ , then, by stability, it is definable over  $M$ , and  $p$  is the unique non-forking extension of  $p|_M$ . Then for any  $\sigma \in \text{Aut}(\mathfrak{C})$ ,  $\sigma(p)$  is the unique non-forking extension of  $\sigma(p|_M)$ . So if  $\sigma(p|_M) = p|_M$ , this means that  $\sigma(p) = p$  so  $p$  does not splinter over  $M$ .  $\square$

*Claim 5.3.* Outside of the stable context, splitting  $\neq$  splintering.

*Proof.* Let  $T$  be the theory of random graphs in the language  $\{I\}$ . Let  $M \models T$  be countable, and let  $a \neq b \in M$  with an automorphism  $\sigma \in \text{Aut}(M)$  taking  $a$  to  $b$ . Let  $p(x) \in S(\mathcal{C})$  say that  $xIc$  for every  $c \in M$  and if  $c \notin M$  then  $xIc$  if and only if  $c$  is connected to  $a$  and not connected to  $b$ . Obviously,  $p$  does not split over  $M$ . However, let  $\sigma' \in \text{Aut}(\mathcal{C})$  be an extension of  $\sigma$ . Let  $c \in \mathcal{C}$  be such that  $c$  is connected to  $a$  but not to  $b$ . Then  $xIc \in p$  but  $xIc \notin \sigma'(p)$ .  $\square$

However,

*Claim 5.4.* If  $T = \text{Th}(\mathbb{Q}, <)$ , then splitting equals splintering.

*Proof.* Observe that by quantifier elimination every complete type  $r(x_0, \dots, x_{n-1})$  over a set  $A$  is determined by  $\text{tp}(x_0, \dots, x_{n-1}/\emptyset) \cup \bigcup \{\text{tp}(x_i/A) \mid i < n\}$ . Assume  $q(x_0, \dots, x_{n-1})$  is a global type that splinters but does not split over a model  $M$ . Then it follows that for some  $i < n$ ,  $q \upharpoonright x_i$  splinters, so we may assume  $n = 1$ . Suppose  $\sigma \in \text{Aut}(\mathcal{C})$  is such that  $\sigma(M) = M$ ,  $\sigma(q|_M) = q|_M$  and  $\sigma(q) \neq q$ . Note that  $\sigma(q^{(\omega)}) = \sigma(q)^{(\omega)}$ , so by Fact 4.10,  $\sigma(q^{(\omega)})|_M \neq q^{(\omega)}|_M$ . We get a contradiction by quantifier elimination again.  $\square$

We shall now generalize Claim 5.3 to every theory with the independence property. In fact, to any theory with large or medium directionality.

**Theorem 5.5.** *Suppose  $T$  has medium or large directionality then splitting  $\neq$  splintering.*

*Proof.* We know that there is some  $p$  and  $\Delta$  such that  $\text{uf}_\Delta(p)$  is infinite. Let us use Construction 4.5:

We may find a saturated model of  $\text{Th}(M^*)$  of size  $\lambda$  where  $\lambda$  is big enough. Then there is  $c, d \in Q_0$  such that  $\text{tp}(c/\emptyset) = \text{tp}(d/\emptyset)$  in the extended language (with symbols for  $N_0, M_0, Q_0$  and  $\bar{f}$ ). So there is an automorphism  $\sigma$  of this structure (in particular of  $N'_0$ ) such that  $\sigma(c) = d$ . By definition,  $\sigma(N_0) = N_0$  and  $\sigma(M_0) = M_0$ . So  $\text{tp}(c/N_0)$  is finitely satisfiable in  $M_0$  and hence does not split over  $M_0$ . But it splinters since  $\sigma(\text{tp}(c/M_0)) = \text{tp}(d/M_0) = \text{tp}(c/M_0)$  but  $\sigma(\text{tp}(c/N_0)) = \text{tp}(d/N_0) \neq \text{tp}(c/N_0)$  as witnessed by  $\varphi$ .

If there are no saturated models, we can take a big enough special model (see [Hod93, Theorem 10.4.4]).

Note that we may also find an example of a type  $p \in S(M)$  with a splintering, non-splitting, global extension, with  $|M| = |T|$ : consider the structure  $(N'_0, N_0, M_0, \sigma, c, d)$ , and find an elementary substructure of size  $|T|$ .  $\square$

**Definition 5.6.** Let  $T$  be a complete theory. We say that  $(M, p, \varphi(x, y), A_1, A_2)$  is an *sp-example* for  $T$  when

- $M \models T$ ;  $A_1, A_2 \subseteq M$  are nonempty and disjoint;  $p = p(x)$  is a complete type over  $M$ , finitely satisfiable in  $A_1$ ;  $\text{Th}(M_p, A_1) = \text{Th}(M_p, A_2)$  (see Definition 4.16); For each pair of finite sets  $s_1 \subseteq A_1$  and  $s_2 \subseteq A_2$ ,  $M \models \exists y (\bigwedge_{a \in s_1} \varphi(a, y) \wedge \bigwedge_{b \in s_2} \neg \varphi(b, y))$ .

**Proposition 5.7.**  *$T$  has an sp-example if and only if there is a finitely satisfiable type over a model which splinters over it  $T$  (in particular, splitting is different than splinting).*

*Proof.* Suppose  $(M, p, \varphi(x, y), A_1, A_2)$  is an sp-example for  $T$ . Let  $M'$  be the structure  $(M_p, A_1, A_2)$  (in the language  $L_p \cup \{P_1, P_2\}$  where  $P_1, P_2$  are predicates). Assume  $|T| < \mu = \mu^{<\mu}$ , and let  $N' = (N_q, B_1, B_2)$  be a saturated extension of  $M'$  of size  $\mu$  where  $N = N' \upharpoonright L$  and  $q = q^N$  is as in Remark 4.17. Since  $(N_q, B_1) \equiv (N_q, B_2)$ , there is an automorphism  $\sigma$  of  $N_q$ , such that  $\sigma$  takes  $B_1$  to  $B_2$  and so  $\sigma(q) = q$ . Let  $q'$  be a global extension of  $q$ , finitely satisfiable in  $B_1$  and  $\sigma'$  a global extension of  $\sigma$ .

So  $q'$  does not split over  $N$ , but it splinters:

Consider the type  $\{\varphi(a, y) \mid a \in B_1\} \cup \{\neg \varphi(b, y) \mid b \in B_2\}$ . It is finitely satisfiable in  $N$  by choice of  $\varphi$ . Let  $c \in \mathfrak{C}$  satisfy this type. Then  $\varphi(x, c) \in q'$  but  $\varphi(x, c) \notin \sigma(q')$  (because  $\sigma(q')$  is finitely satisfiable in  $B_2$ ).

If we do not assume the existence of such a  $\mu$ , we can use special models.

Now suppose that splitting is different than splintering, as witnessed by some global type  $p$  that splinters over a model  $M$  but is finitely satisfiable in it. Then there is some automorphism  $\sigma$  of  $\mathfrak{C}$  that witnesses it. There is a formula  $\varphi(x, y)$  and  $a \in \mathfrak{C}$  such that  $\varphi(x, a) \in p, \neg \varphi(x, a) \in \sigma(p)$ . Let  $B_1 = \{m \in M \mid \varphi(m, a) \wedge \neg \varphi(m, \sigma^{-1}(a))\}$ ,  $B_2 = \sigma(B_1)$ . It is easy to check that  $(M, p, \varphi, B_1, B_2)$  is an sp-example  $\square$

The following theorem answers the natural question:

**Theorem 5.8.** *There is a theory with small directionality in which splitting  $\neq$  splintering.*

*Proof.* Let  $L = \{R\}$  where  $R$  is a ternary relation symbol. Let  $M_0 = \langle \mathbb{Q}, < \rangle$  and define  $R(x, y, z)$  by  $x < y < z$  or  $z < y < x$ , i.e.  $y$  is between  $x$  and  $z$ . Let  $T = \text{Th}(M_0 \upharpoonright L)$ .

*Claim.*  $T$  has small directionality.

*Proof.* Suppose  $M \models T$ . Let  $(a, b)$  denote  $\{c \mid R(a, c, b)\}$ .

Then, for any choice of a pair of distinct elements  $a, b$  there is a unique enrichment of  $M$  to a model  $M'$  of  $\text{Th}(\mathbb{Q}, <)$  such that  $a < b$ :

For  $w \neq z$ ,  $w < z$  if and only if

$(a, b) \cap (a, z) \neq \emptyset$  and  $((a, w) \subseteq (a, z) \text{ or } (a, w) \cap (a, z) = \emptyset)$  or

$(a, b) \cap (a, z) = \emptyset$  and  $(z, b) \subseteq (w, b)$ .



From this observation, it follows that there is a unique completion of any type  $p \in S(M)$  to a type  $p' \in S(M')$ . So if  $\Delta$  is a finite set of  $L$  formulas and  $uf_\Delta(p)$  is infinite, then  $uf_\Delta(p')$  is also infinite — contradiction to Example 4.24.  $\square$

*Claim.*  $T$  has an sp-example

*Proof.* Let  $M = M_0 \upharpoonright L$ . Let  $p(x) = tp(\pi/M)$ . Let  $A_1 = \{x \in \mathbb{Q} \mid x > \pi\}$  and  $A_2 = \mathbb{Q} \setminus A_1$ , and let  $\varphi(x, y_1, y_2) = R(y_1, x, y_2)$ . We claim that  $(M, p, \varphi(x, y_1, y_2), A_1, A_2)$  is an sp-example:

First, let  $M'$  be the reduct of  $(\mathbb{Q} \cup \{\pi\}, <)$  to  $L$ . There is some  $\sigma \in \text{Aut}(M'/\pi)$  such that  $\sigma(A_1) = A_2$ . Hence  $(M_p, A_1) \cong (M_p, A_2)$ . Also, since  $tp(\pi/M_0)$  (in  $\{<\}$ ) is finitely satisfiable in both  $A_1$  and  $A_2$  (by quantifier elimination),  $p$  is finitely satisfiable in both  $A_1$  and  $A_2$ . Finally, for finite  $s_1 \subseteq A_1$  and  $s_2 \subseteq A_2$ , there exists  $c_1, c_2 \in \mathbb{Q}$  such that  $R(c_1, a, c_2)$  for all  $a \in s_1$  and  $\neg R(c_1, b, c_2)$  for all  $b \in s_2$ .  $\square$

$\square$

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